Institut für angewandte Mathematik
Sommersemester 2018
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„Stochastic Processes", Problem Sheet 12.

Please hand in your solutions before 3 pm on Tuesday, July 10.

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1. (Transition probabilities for continuous-time Markov chains I).

a) Compute $p_t(1,1)$ for the Markov process with state space $S = \{1,2,3\}$ and generator

$$\mathcal{L} \; = \begin{pmatrix} -2 & 1 & 1 \\ 4 & -4 & 0 \\ 2 & 1 & -3 \end{pmatrix}.$$

b) Which of the following matrices are exponentials of Q-matrices ?

$$(i) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (ii) \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \qquad (iii) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

2. (Transition probabilities for continuous-time Markov chains II). Two fleas are bound together to take part in a nine-legged race on the vertices A, B, C of a triangle. Flea 1 hops at random times in the clockwise direction; each hop takes the pair from one vertex to the next and the times between successive hops of Flea 1 are independent random variables, each with exponential distribution, mean $1/\lambda$. Flea 2 behaves similarly, but hops in the anticlockwise direction, the times between his hops having mean $1/\mu$. Show that the probability that they are at A at a given time $t > 0$ (starting from A at time $t = 0$) is

$$\frac{1}{3} + \frac{2}{3} \exp\left(-\frac{3(\lambda+\mu)t}{2}\right) \cos\left(\frac{\sqrt{3}(\lambda-\mu)t}{2}\right). \tag{1}$$

3. (Wiener–Lévy representation and quadratic variation).
The quadratic variation $[x]_t$ of a continuous function $x : \mathbb{R}_+ \to \mathbb{R}$ along the sequence of dyadic partitions of the intervals $[0,t]$ is defined by

$$[x]_t : = \lim_{m\to\infty} \sum_{i=1}^{2^m} \left| x(t_i^{(m)}) - x(t_{i-1}^{(m)}) \right|^2 ; \qquad t_i^{(m)} = i 2^{-m} t.$$

a) Show that the quadratic variation of a continuously differentiable function x vanishes, i.e., $[x]_t = 0$ for any $t \geq 0$.

b) Let

$$x(t) \;= x(1)\cdot t\,+\,\sum_{n=0}^{\infty}\sum_{k=0}^{2^n-1} a_{n,k}\cdot e_{n,k}(t), \qquad a_{n,k}\in\mathbb{R},$$

be the expansion of a function $x \in C([0,1])$ with $x(0) = 0$ in the basis of Schauder functions. Show that

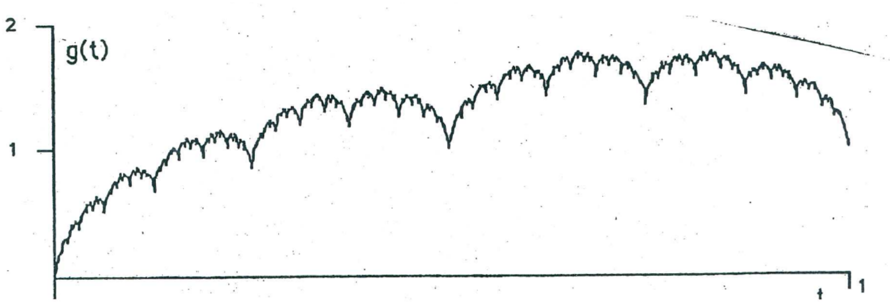
$$[x]_1 \;=\; \lim_{m\to\infty}\frac{1}{2^m}\sum_{n=0}^{m-1}\sum_{k=0}^{2^n-1}(a_{n,k})^2.$$

c) Deduce that almost every path of Brownian motion has quadratic variation $[B]_t = t$. Why does it suffice to consider $t = 1$?

d) Determine the quadratic variation of the "self-similar" function

$$g(t) \;:=\; t\,+\,\sum_{n=0}^{\infty}\sum_{k=0}^{2^n-1} e_{n,k}(t)$$

on the interval $[0,1]$, and on $[0,t]$ for $t\in[0,1)$. Compare with the results from b) and c).



4. (Stopped Brownian motion). Let X_t be a one-dimensional Brownian motion started at 0 and let $T = \min\{t : |X_t| = 1\}$ and $T^* = \min\{t : X_t = 1 \text{ or } X_t = -3\}$.

a) Explain why X_T and T are independent random variables.

b) Show that T^* and X_{T*} are not independent.