Institut für angewandte Mathematik
Sommersemester 2018
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„Stochastic Processes", Problem Sheet 11.

Please hand in your solutions before 3 pm on Tuesday, July 3.

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1. (Symmetries of Brownian motion).

Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be independent one-dimensional Brownian motions starting at 0.

a) Show that the following processes are Brownian motions:

$$\text{(i)} \ -X_t \qquad \text{(ii)} \ X_{t+h} - X_h \ \ (h \geq 0 \text{ fixed}) \qquad \text{(iii)} \ \frac{1}{\sqrt{a}} X_{at} \ \ (a > 0 \text{ fixed}).$$

b) Show that $B_t := (X_t - Y_t)/\sqrt{2}$ is a Brownian motion.

c) True or false: With probability 1 we have $X_t = Y_t$ for infinitely many $t > 0$.

2. (Wiener-Lévy Representation of Brownian Motion).

The Schauder functions $e_{n,k} \in C([0,1])$ are defined in the following way :

$$\begin{aligned} e_{0,0}(t) &:= \min(t, 1-t), \\ e_{n,k}(t) &:= \begin{cases} 2^{-n/2} \cdot e_{0,0}(2^n t - k) & \text{for } t \in [k2^{-n}, (k+1)2^{-n}], \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

$n \in \mathbb{N}$, $k = 0, 1, 2, \dots, 2^n - 1$. For $x \in C([0,1])$ with $x(0) = 0$ let

$$a_{n,k} \ := \ -2^{n/2} \cdot \Delta_{n,k} x \qquad \text{with} \qquad \Delta_{n,k} x := 2(\bar{x}_{n,k} - x(m_{n,k})),$$

where $\bar{x}_{n,k} := (x((k+1)\cdot 2^{-n}) + x(k \cdot 2^{-n}))/2$ and $m_{n,k}$ denotes the midpoint of the dyadic interval $[k2^{-n}, (k+1)2^{-n}]$. Show that :

a) The sequence

$$x^{(m)}(t) \ := \ x(1) \cdot t + \sum_{n=0}^{m} \sum_{k=0}^{2^n - 1} a_{n,k} \cdot e_{n,k}(t), \qquad m \in \mathbb{N},$$

converges to $x(t)$ uniformly for $t \in [0,1]$. *(Hint : Verify that $x^{(m)}$ is the polygonal approximation of x w.r.t. the m–th dyadic partition of the interval $[0,1]$)*

b) Under Wiener measure μ_0 on $\Omega = C([0,1])$, the random variables

$$X_1(\omega) \qquad \text{und} \qquad Z_{n,k}(\omega) \;:=\; -2^{n/2} \cdot \Delta_{n,k} X(\omega) \qquad (n \geq 0, 0 \leq k < 2^n),$$

are independent with distribution $N(0,1)$, and the *Wiener–Lévy Representation*

$$X_t(\omega) \;=\; X_1(\omega) \cdot t \,+\, \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} Z_{n,k}(\omega) \cdot e_{n,k}(t) \qquad \text{holds for all } \omega \in \Omega.$$

c) How can this be used in order to simulate sample paths of Brownian motion ?

3. (Local maxima of Brownian paths). Let B_t be a one-dimensional Brownian motion on $(\Omega, \mathcal{A}, P)$. Show that the following statements hold for almost every ω:

a) The trajectory $t \mapsto B_t(\omega)$ is not monotone in any interval $[a,b]$ with $a < b$.

b) The set of local maxima of $t \mapsto B_t(\omega)$ is dense in $[0, \infty)$.

c) All local maxima of $t \mapsto B_t(\omega)$ are strict (i.e., for any local maximum m there exists an $\varepsilon > 0$ such that $B_t(\omega) < B_m(\omega)$ for all $t \in (m - \varepsilon, m + \varepsilon)$).

4. (Transformations of Brownian motion).

a) Show that the projection of a d-dimensional Brownian motion onto a line through the origin yields a one-dimensional Brownian motion: Suppose that $(W_t^{(1)}, \ldots, W_t^{(d)})$ is a d-dimensional Brownian motion starting from 0, and let $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$ with $\sum_{i=1}^d \lambda_i^2 = 1$. Show that $X_t = \sum_{i=1}^d \lambda_i W_t^{(i)}$ is a Brownian motion starting from 0.

b) Show that rotating a Brownian motion about the origin yields another Brownian motion: Let W be a d-dimensional Brownian motion starting from 0 and let A be a $d \times d$ orthogonal matrix. Show that $Y_t = AW_t$ is again a d-dimensional Brownian motion.

c) Formulate a statement that includes a) and b) !