Institut für angewandte Mathematik
Sommersemester 2018
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„Stochastic Processes", Problem Sheet 10.

Please hand in your solutions before 3 pm on Tuesday, June 26.

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1. (Probabilities of Brownian motion).
Let B_t $(t \geq 0)$ be a one-dimensional Brownian motion on $(\Omega, \mathcal{A}, P)$ with $B_0 = 0$.

a) Compute the probabilities of the following events:

  (i) $B_2 > 2$.

  (ii) $B_2 > B_1$.

  (iii) $B_2 > B_1 > B_3$.

b) Let $Z := \sup_{t \geq 0} B_t$. Show that λZ has the same distribution as Z for any $\lambda > 0$. Deduce that $Z = +\infty$ P–a.s.

2. (Martingales of Brownian motion). State the definition of a martingale in continuous time. Show that the following processes are martingales:

a) A one-dimensional Brownian motion $(B_t)_{t \geq 0}$ w.r.t. $\mathcal{F}_t = \sigma(B_r : 0 \leq r \leq t)$.

b) $M_t^\lambda = \exp\left(\lambda B_t - \frac{1}{2}\lambda^2 t\right)$, $\lambda \in \mathbb{R}$, w.r.t. the same filtration.

c) $h(B_t)$, if $(B_t)_{t \geq 0}$ is a d–dimensional Brownian motion, and h is a harmonic function on $\mathbb{R}^d$, w.r.t. $\mathcal{F}_t = \sigma(B_r^1, B_r^2, \ldots, B_r^d \,:\, 0 \leq r \leq t)$.

3. (Zeros of Brownian paths).
Let $(B_t)_{t \in [0,1]}$ be a Brownian motion on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ whose sample paths $t \mapsto B_t(\omega)$ are all continuous.

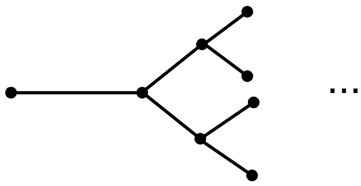
a) Show that $(t, \omega) \mapsto B_t(\omega)$ is measureable as a map from $\Omega \times [0,1]$ to $\mathbb{R}$.

b) Compute the expectation and the variance of $\int_0^1 B_s(\omega)\, ds$.

c) Show that $\mathbb{P}$–a.s.: $\lambda\left[\{t \in [0,1] : B_t = 0\}\right] = 0\,.$

4. (Recurrence and transience of random walks).

a) Is the simple random walk on the infinite binary tree



recurrent or transient (see Exercise 8.3) ?

b) Prove that: For the classical random walk on $\mathbb{Z}^3$ we have

$$p^{2n}(x,x) \;=\; \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \sum_{\substack{i,j,k\geq 0\\ i+j+k=n}} \binom{n}{i\,j\,k}^2 \left(\frac{1}{3}\right)^{2n} \;\leq\; \frac{const.}{n^{3/2}}\,,$$

where $\binom{n}{i\,j\,k} := \frac{n!}{i!j!k!}$. Deduce that the random walk is transient.

Hint : $\sum_{i+j+k=n} \binom{n}{i\,j\,k} = 3^n$.