Institut für Angewandte Mathematik
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“Stochastic Analysis”, Problem Sheet 9

Please hand in your solutions by Wednesday, 1 July, 12:00.

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1. (Couplings on $\mathbb{R}^d$). Let $W\colon \Omega \to \mathbb{R}^d$ be a random variable on $(\Omega, \mathcal{A}, \mathbb{P})$, and let μ_a denote the law of $a + W$.

a) Show that for $a, b \in \mathbb{R}^d$, every coupling of μ_a and μ_b can be realised by random variables $X = a + W_1$ and $Y = b + W_2$ with $W_1, W_2 \sim W$.

b) (Synchronous coupling) Let $X = a + W$ and $Y = b + W$. Show that (X, Y) is an optimal coupling of μ_a and μ_b w.r.t. $\mathcal{W}^2$.

c) More generally, show that synchronous coupling is optimal w.r.t. $\mathcal{W}^p$ for any $p \in [1, \infty)$.

d) (Reflection coupling) Now assume that the law of W is invariant under orthogonal transformations of $\mathbb{R}^d$, and let $\widetilde{Y} = b + \widetilde{W}$ where $\widetilde{W} = W - 2(e^T W)e$ with $e = \frac{a-b}{|a-b|}$. Prove that $(X, \tilde{Y})$ realises a coupling of μ_a and μ_b, and if $|W| \leq \frac{|a-b|}{2}$ a.s., then

$$\mathbb{E}\left[f(|X - \widetilde{Y}|)\right] \;\leq\; f(|a-b|) \;=\; \mathbb{E}\left[f(|X - Y|)\right]$$

for any concave, increasing function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = 0$.

2. (Distances between normal distributions). Let $\mu = \mathcal{N}(a, \sigma^2)$ and $\nu = \mathcal{N}(b, \sigma^2)$ be one-dimensional normal distributions with means $a, b \in \mathbb{R}$ and standard deviation $\sigma \in (0, \infty)$.

a) Show that for any $p \in [1, \infty)$,

$$\mathcal{W}^p(\mu, \nu) \;=\; |a - b|\,.$$

b) Show that

$$\mathrm{TV}(\mu, \nu) \;=\; 2\Phi\left(\frac{|a-b|}{2\sigma}\right) - 1\,.$$

Conclude that asymptotically as $|a - b| \to 0$,

$$\mathrm{TV}(\mu, \nu) \;\sim\; \frac{|a-b|}{\sqrt{2\pi\sigma^2}}\,.$$

How does $\mathrm{TV}(\mu, \nu)$ behave asymptotically as $|a - b| \to \infty$?

c) Also compute $\mathcal{W}_f(\mu, \nu)$ for

$$f(r) \;=\; \sqrt{2\pi\sigma^2}\,1_{(0,\infty)}(r) + r\,.$$

Plot the different distances as a function of $|a - b|$.

3. (Simulation of stochastic flows). We consider a stochastic process $X_t = (X_{t,1}, X_{t,2})^T$ with values in $\mathbb{R}^2$ that solves a stochastic differential equation

$$\mathrm{d}X_t \;=\; AX_t\,\mathrm{d}t \,+\, CX_t^3\,\mathrm{d}t \,+\, \sigma\,\mathrm{d}B_t, \quad X_0 = x_0,$$

where $(B_t)_{t\geq 0}$ is a two dimensional Brownian motion, A and C are 2×2 matrices, $\sigma \in (0,\infty)$ and the initial value $x_0 \in \mathbb{R}^2$ are given constants, and $X_t^3 := (X_{t,1}^3, X_{t,2}^3)^T$.

a) Write down and implement the Euler-Maruyama discretization of the SDE over a time interval $[0, t_{\max}]$ for a given time step size $h > 0$.

b) We consider at first the case of a linear drift, i.e., $C = 0$. In this case, the solution of the SDE is a Gaussian process. Simulate and plot sample paths for the following choices of A and σ, $t_{\max} = 400$ and $x_0 = (1,0)^T$. To obtain sufficiently accurate approximations choose $h = 0.001$ if possible on your computer, at least $h \leq 0.01$.

   (i) *Two dimensional Brownian motion:* $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\sigma = 1$.

   (ii) *Standard two dimensional OU process:* $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma = 1$.

   (iii) *Randomly perturbed rotation:* $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\sigma = 0.1$ and $\sigma = 1$.

   (iv) *Randomly perturbed rotation with damping:* $A = \begin{pmatrix} -\gamma & 1 \\ -1 & -\gamma \end{pmatrix}$, $\sigma = \gamma \in \{0.1, 1\}$.

   (v) *Perturbed Langevin dynamics:* $A = \begin{pmatrix} 0 & 1 \\ -1 & -\gamma \end{pmatrix}$, $\sigma = \sqrt{2\gamma}$, $\gamma \in \{0.1, 1\}$.

c) Visualise the stochastic flows corresponding to the examples above by plotting in a single graphic the trajectories of solutions for a family of nearby initial conditions.

d) Now consider the overdamped Langevin process solving the SDE with parameters

   (vi) *OL in double-well potential:* $A = \begin{pmatrix} -0.625 & 1.25 \\ 1.25 & 17.5 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 0 & -20 \end{pmatrix}$, $\sigma = \sqrt{2}$.

   Compute the Euler-Maruyama approximation with step size $h = 0.001$ and initial value $x_0 = (0,0)^T$ up to $t_{\max} = 15$, and with step size $h = 0.01$ and the ergodic averages up to $t_{\max} = 1000$.

e) Again, visualise the corresponding stochastic flow for initial values close to $x_0 = (0,0)^T$.

Hint: It might make sense to choose a higher resolution and thin lines for the plots.
For example in Python if the numerical solution is stored in a $2 \times$ steps *array* sde:

```
plt.figure(figsize=(7,7), dpi=500)
plt.plot(sde[0],sde[1],linewidth=.1)
plt.show()
```

Matrix multiplication in Python: `np.matmul(A,B)`