Institut für Angewandte Mathematik
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"Stochastic Analysis", Problem Sheet 8

Please hand in your solutions by Wednesday, 24 June, 12:00.

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1. (Brownian local time). Let L^y denote the local time in y of a one-dimensional Brownian motion $(B, \mathbb{P}_x)$ starting at $x \in \mathbb{R}$.

a) Let $T = \inf\{t \geq 0 : B_t \notin (a, b)\}$ with $a \leq x \leq y \leq b$. Show that

$$\mathbb{E}_x[L_T^y] \;=\; \frac{2(x-a)(b-y)}{b-a}.$$

b) Show that for all $a > 0$, the processes $(L_{at}^0)_{t\geq 0}$ and $(\sqrt{a}L_t^0)_{t\geq 0}$ have the same law under $\mathbb{P}_0$.

2. (Some applications of Burkholder's inequality). Let $(W, \mathbb{P})$ be a one-dimensional Brownian motion starting at 0.

a) Suppose that X is a solution to an SDE

$$dX_t \;=\; \sigma(X_t)\, dW_t \;+\; b(X_t)\, dt, \qquad X_0 = x_0,$$

where $x_0 \in \mathbb{R}$, and σ and b are continuous real-valued functions such that

$$|\sigma(x)| + |b(x)| \;\leq\; c\,(1 + |x|) \qquad \text{for some } c \in (0, \infty).$$

Show that for every $t_0 \in [0, \infty)$ and $p \in [2, \infty)$, there is a finite constant C such that

$$\mathbb{E}\left[\sup_{s\leq t_0} |X_s|^p\right] \;\leq\; C\;(1 + |x_0|^p)\,.$$

b) Show that $\mathbb{E}[W_T] = 0$ holds for any stopping time T such that $\mathbb{E}[T^{1/2}] < \infty$.

3. (Burkholder-Davis-Gundy revisited). This exercise contains an alternative proof of the upper bound in the BDG inequality. Let $p > 0$.

a) Suppose X and Y are non-negative random variables such that, for some $\beta > 1$, $\delta \in (0,1)$ and $\varepsilon \in (0, \beta^{-p}/2)$, we have

$$\mathbb{P}[X > \beta\lambda, Y < \delta\lambda] \ \leq \ \varepsilon \, \mathbb{P}[X \geq \lambda] \qquad \text{for all } \lambda > 0.$$

Show that there exists a finite constant C_p (depending only on p, β, δ and ε but not on X and Y) such that

$$\mathbb{E}[X^p] \ \leq \ C_p \, \mathbb{E}[Y^p].$$

b) Let $(B, \mathbb{P})$ be a Brownian motion starting at 0. Use part (i) to show Burkholder's inequality for B. That is, show that there exists a finite constant C_p such that for all finite stopping times T,

$$\mathbb{E}\left[(B_T^*)^p\right] \ \leq \ C_p \, \mathbb{E}[T^{p/2}].$$

Hint. Let $U = \inf\{t \geq 0 : |B_t| > \lambda\}$. *Show first that*

$$\mathbb{P}[B_T^* > \beta\lambda, T^{1/2} < \delta\lambda] \leq \mathbb{P}\left[\sup_{U \leq t \leq U + \delta^2\lambda^2} |B_t - B_U| > (\beta - 1)\lambda, \, U < \infty\right].$$

c) Now conclude from part b) that the upper bound in the BDG inequality holds for an arbitrary continuous local martingale $(M, \mathbb{P})$ with $M_0 = 0$.