Institut für Angewandte Mathematik
Summer semester 2026
Andreas Eberle, Francis Lörler

“Stochastic Analysis”, Problem Sheet 6

Please hand in your solutions by Wednesday, 10 June, 12:00.

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1. (SDE with linear coefficients). Consider the SDE

$$dX_t \;=\; A_t X_t\,dt \,+\, \sum_{k=1}^{d} \sigma_t^k X_t\,dB_t^k, \tag{1}$$

where (B,P) is a d-dimensional Brownian motion, and $A, \sigma^1, \ldots, \sigma^d$ are $(\mathcal{F}_t^{B,P})$ adapted, bounded, $(n\times n)$ matrix-valued continuous processes.

a) Show that for a given initial value $a \in \mathbb{R}^n$, the equation has a unique strong solution.

b) Determine the solution in the case $n = 1$ explicitly.

c) Suppose that $X = (X_t)$ is a fundamental solution of (1), i.e., X is an $(n\times n)$ matrix-valued process that satisfies (1) with initial condition $X_0 = I_n$. Show that almost surely, X_t is an invertible matrix for every t, and the inverse process $Z_t = X_t^{-1}$ satisfies

$$dZ_t \;=\; Z_t\left(\sum_{k=1}^{d}\left(\sigma_t^k\right)^2 - A_t\right)dt \,-\, \sum_{k=1}^{d} Z_t\sigma_t^k\,dB_t^k, \qquad Z_0 \;=\; I_n. \tag{2}$$

Hint: Define Z as the solution of (2), and verify that $Z = X^{-1}$ almost surely.

2. (Stratonovich to Itô conversion).

a) Show that the “associative law”

$$\int X \circ d\left(\int Y \circ dZ\right) \;=\; \int XY \circ dZ$$

holds for Stratonovich integrals of continuous semimartingales X, Y, Z.

b) We consider a Stratonovich SDE in $\mathbb{R}^n$ of the form

$$\circ dX_t \;=\; b(X_t)\,dt + \sum_{k=1}^{d} \sigma_k(X_t)\circ dB_t^k, \qquad X_0 \;=\; x_0, \tag{3}$$

with $x_0 \in \mathbb{R}^n$, vector fields $b \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $\sigma_1, \ldots, \sigma_d \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, and an $\mathbb{R}^d$-valued Brownian motion (B_t). Prove that (3) is equivalent to the Itô SDE

$$dX_t \;=\; \tilde{b}(X_t)\,dt + \sum_{k=1}^{d} \sigma_k(X_t)\,dB_t^k, \qquad X_0 \;=\; x_0, \tag{4}$$

where $\tilde{b} := b + \frac{1}{2}\sum_{k=1}^{d} \sigma_k \cdot \nabla\sigma_k$. Conclude that if $\tilde{b}$ and $\sigma_1, \ldots, \sigma_d$ are Lipschitz continuous, then there is a unique strong solution of (3).

3. (Brownian motion on hypersurfaces). Let $f \in C^\infty(\mathbb{R}^{n+1})$ such that $f(x) \to \infty$ as $|x| \to \infty$, and suppose that $c \in \mathbb{R}$ is a regular value of f, i.e., $\nabla f(x) \neq 0$ for any $x \in f^{-1}(c)$. Then by the implicit function theorem, the level set

$$M_c \quad = \quad f^{-1}(c) \quad = \quad \{x \in \mathbb{R}^{n+1} \; : \; f(x) = c\}$$

is a smooth compact n-dimensional submanifold of $\mathbb{R}^{n+1}$. For $x \in M_c$, the vector

$$\mathfrak{n}(x) \quad = \quad \nabla f(x)/|\nabla f(x)| \quad \in S^n$$

is the **unit normal** to M_c at x. The **tangent space** to M_c at x is the orthogonal complement

$$T_x M_c \quad = \quad \mathrm{span}\{\mathfrak{n}(x)\}^\perp.$$

Let $P(x) : \mathbb{R}^{n+1} \to T_x M_c$ denote the orthogonal projection onto the tangent space w.r.t. the Euclidean metric, i.e.,

$$P(x)v \quad = \quad v - v \cdot \mathfrak{n}(x)\, \mathfrak{n}(x), \quad v \in \mathbb{R}^{n+1}.$$

For $k \in \{1, \ldots, n+1\}$, we set $P_k(x) = P(x)e_k$. A **Brownian motion on the hypersurface $\boldsymbol{M_c}$** with initial value $x_0 \in M_c$ is a solution (X_t) of the Stratonovich SDE

$$\circ dX_t \quad = \quad P(X_t) \circ dB_t \quad = \quad \sum_{k=1}^{n+1} P_k(X_t) \circ dB_t^k, \qquad X_0 \; = \; x_0, \tag{5}$$

with respect to a Brownian motion (B_t) on $\mathbb{R}^{n+1}$.

a) Prove that almost surely, there exists a unique strong solution $(X_t)_{t \in [0,\infty)}$ of (5), and (X_t) stays on the submanifold M_c for all times.

b) Prove that the SDE (5) can be written in Itô form as

$$dX_t \quad = \quad P(X_t)\; dB_t - \frac{n}{2}\, \kappa(X_t)\, \mathfrak{n}(X_t)\; dt$$

where $\kappa(x) = \frac{1}{n} \operatorname{div} \mathfrak{n}(x)$ is the mean curvature of M_c at x.

Hint: Note that $\mathfrak{n} \cdot \nabla \mathfrak{n} = 0$ *and* $\mathfrak{n} \cdot \nabla P_k + P_k \cdot \nabla \mathfrak{n} = 0$. *Why do these identities hold ?*