Institut für Angewandte Mathematik
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"Stochastic Analysis", Problem Sheet 5

Please hand in your solutions by Wednesday, 3 June, 12:00.

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1. (Brownian scale invariance). Let $\sigma \in C(\mathbb{R}^d, \mathbb{R}^{d\times d})$ and $b \in C(\mathbb{R}^d, \mathbb{R}^d)$ satisfying

$$\sigma(cx) = \sigma(x) \qquad \text{and} \qquad c\, b(cx) = b(x) \qquad \forall c > 0,\ x \in \mathbb{R}^d.$$

Prove that if uniqueness in law holds for the SDE

$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t + b(X_t)\,\mathrm{d}t,$$

and if X_t^a is a weak solution with $X_0^a = a$, then for any $c > 0$, the processes

$$\left(c^{-1}X^a_{c^2t}\right)_{t\geq 0} \qquad \text{and} \qquad \left(X_t^{a/c}\right)_{t\geq 0}$$

have the same law. Give examples for processes having this scale invariance.

2. (Overdamped Langevin dynamics and Brownian motion with killing). Suppose that $(X, \mathbb{P}_x)$ is a solution of the stochastic differential equation

$$\mathrm{d}X_t \;=\; -\nabla U(X_t)\,\mathrm{d}t \;+\; \mathrm{d}B_t\,, \qquad X_0 = x\,, \tag{1}$$

where $U \in C^2(\mathbb{R}^d, [0,\infty))$, and B is a d-dimensional Brownian motion under $\mathbb{P}_x$. Let

$$V(x) \;:=\; \frac{1}{2}\,|\nabla U(x)|^2 \;-\; \frac{1}{2}\Delta U(x)\,,$$

and $V_0(x) := V(x) - a$ where $a = \inf V$. We assume $a > -\infty$.

a) Prove that (1) has a non-explosive weak solution which is a Markov process.

b) Show that the transition function p_t of this process is related to the transition function $p_t^{V_0}$ of Brownian motion with killing rate $V_0(x)$ in the following way: For any measurable function $f\colon \mathbb{R}^d \to [0,\infty)$,

$$e^{ta}\,p_t f \;=\; e^U p_t^{V_0}(e^{-U}f)\,.$$

c) How is the generator $\mathcal{L}$ of (1) related to the Schrödinger operators $\mathcal{L}^{V_0} = \frac{1}{2}\Delta - V_0$ and $\mathcal{L}^V = \frac{1}{2}\Delta - V$? What does this imply for the spectrum of $\mathcal{L}$?

Remark: The ground state transformation of quantum mechanics maps the Schrödinger operator $\mathcal{L}^V$ to $\mathcal{L}$. The transformation "OLD $\mapsto$ BM with killing" *gives a stochastic interpretation of the inverse ground state transform.*

3. (Geometric Poisson processes and change of measure). Let $(N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$ on a filtered probability space $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t))$.

a) Let $\sigma, \alpha \in \mathbb{R}$ with $\sigma > -1$. Give a meaning to the SDE

$$\mathrm{d}S_t \;=\; \sigma\, S_{t-}\,\mathrm{d}N_t \;+\; \alpha\, S_t\,\mathrm{d}t\,, \qquad S_0 = 1,$$

and find a solution by the ansatz $S_t = \exp\left(aN_t + bt\right)$.

b) Given σ, for which value of α is (S_t) a martingale?

c) Now let $\mu > 0$. Verify that

$$Z_t \;=\; (\mu/\lambda)^{N_t}\, e^{(\lambda-\mu)t}$$

is an $(\mathcal{F}_t)$ martingale with $\mathbb{E}[Z_t] = 1$ for all t.

d) We define a new probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}_1)$ by

$$\tilde{\mathbb{P}}[A] \;=\; \int_A Z_1\, d\mathbb{P} \qquad \text{for any } A \in \mathcal{F}_1.$$

Verify that $\tilde{\mathbb{E}}[X_t] = \mathbb{E}[X_t Z_t]$ for any $\mathcal{F}_t$-measurable random variable X_t and $t \in [0,1]$. Compute the characteristic function of the process $(N_t)_{t\in[0,1]}$ w.r.t. the new measure $\tilde{\mathbb{P}}$. Conclude that under $\tilde{\mathbb{P}}$, (N_t) is a Poisson process with intensity μ.

4. (Tweedie's formula and score matching). Let μ be a probability measure on $\mathbb{R}^d$.

a) Suppose that $Y = X + \sigma Z$ where X and Z are independent random variables with laws $X \sim \mu$ and $Z \sim \mathcal{N}(0, I_d)$, and σ is a non-degenerate $d \times d$-matrix. We can interpret Y as a noisy observation of X. Show that

$$\mathbb{E}[X \mid Y] \;=\; Y + \sigma\sigma^T \nabla \log \rho(Y)$$

where ρ is the density of the law of Y.

b) Consider a diffusion model with forward in time dynamics given by the SDE

$$\mathrm{d}X_t \;=\; f(t)X_t\,\mathrm{d}t \;+\; g(t)\,\mathrm{d}B_t\,, \qquad X_0 \sim \mu\,, \tag{2}$$

with continuous functions $f, g\colon [0,\infty) \to \mathbb{R}$, $g(t) > 0$, and a Brownian motion B that is independent of X_0. Let $\rho_t(x)$ denote the density of the law of X_t. Give a representation of the *score function*

$$s(x,t) \;=\; \nabla \log \rho_t(x)$$

in terms of conditional expectations.

c) How can this be used to estimate the score function if one is given a large number of samples $(X_t^{(i)})_{t\geq 0}$, $i = 1, \ldots, N$, from the SDE (2)?