

“Stochastic Analysis”, Problem Sheet 4

Please hand in your solutions by Wednesday, 13 May, 12:00.

1. (Passage time to a sloping line). Let $(X_t)_{t \geq 0}$ on $(\Omega, \mathcal{A}, \mathbb{P})$ be a one-dimensional Brownian motion with $X_0 = 0$, and let $a > 0$. Recall that by the reflection principle, the law of the first passage time $T_a = \inf\{t \geq 0 : X_t = a\}$ is absolutely continuous with density

$$f_{T_a}(t) = at^{-3/2} \varphi(a/\sqrt{t}) \mathbf{1}_{(0, \infty)}(t).$$

Here φ denotes the standard normal density.

- a) For $b \in \mathbb{R}$ let $T_L = \inf\{t \geq 0 : X_t = a + bt\}$ denote the first passage time to the line $y = a + bt$. Show that

$$\mathbb{P}[T_L \leq t] = \mathbb{E} \left[e^{-bX_t - b^2 t/2}; T_a \leq t \right] = \int_0^t e^{-ab - b^2 s/2} a s^{-3/2} \varphi(a/\sqrt{s}) ds.$$

Conclude that the law of T_L is absolutely continuous with density

$$f_{T_L}(t) = at^{-3/2} \varphi((a + bt)/\sqrt{t}) \mathbf{1}_{(0, \infty)}(t).$$

- b) Show that for any fixed $b > 0$,

$$\mathbb{E} \left[e^{-bX_t} \max_{s \leq t} X_s \right] \sim \frac{1}{2b} e^{b^2 t/2} \quad \text{asymptotically as } t \rightarrow \infty.$$

2. (Brownian motion writes your name). Prove that Brownian motion in \mathbb{R}^2 will write your name (in cursive script, without dotted i's or crossed t's).

To get the pen rolling, first take $(B_t)_{t \in [0, 1]}$ on $(\Omega, \mathcal{A}, \mathbb{P})$ to be a two-dimensional Brownian motion, and note that for any $[a, b] \subseteq [0, 1]$ the process

$$X_t^{a,b} = (b - a)^{-1/2} (B_{a+t(b-a)} - B_a)$$

is again a Brownian motion on $[0, 1]$. Now, take $g: [0, 1] \rightarrow \mathbb{R}^2$ to be a parametrisation of your name, and note that Brownian motion writes your name (to precision ε) on the interval $[a, b]$ if

$$\sup_{0 \leq t \leq 1} |X_t^{a,b} - g(t)| \leq \varepsilon. \tag{1}$$

- a) Let A_k denote the event that inequality (1) holds for $a = 2^{-k-1}$ and $b = 2^{-k}$. Show that if $\mathbb{P}[A_1] > 0$ then infinitely many of the A_k occur with probability one.

b) Show that

$$\mathbb{P}\left[\sup_{0 \leq t \leq 1} |B_t| \leq \varepsilon\right] > 0.$$

c) Finally, prove that

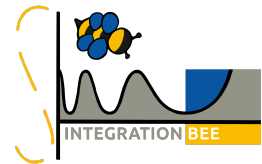
$$\mathbb{P}\left[\sup_{0 \leq t \leq 1} |B_t - g(t)| \leq \varepsilon\right] > 0,$$

and complete the solution of the problem.

3. (Change of measure for continuous semimartingales). Let (\mathcal{F}_t) be a filtration on (Ω, \mathcal{A}) , and let \mathbb{P} and \mathbb{Q} be probability measures that are mutually absolutely continuous on \mathcal{F}_t for any $t \in [0, \infty)$ with densities $Z_t = \frac{d\mathbb{P}}{d\mathbb{Q}} \Big|_{\mathcal{F}_t}$. We assume that (Z_t) is a *continuous* martingale. Show that the following statements hold for an adapted continuous process (X_t) :

- a) X is a martingale w.r.t. \mathbb{P} if and only if $X \cdot Z$ is a martingale w.r.t. \mathbb{Q} .
- b) X is a local martingale w.r.t. \mathbb{P} if and only if $X \cdot Z$ is a local martingale w.r.t. \mathbb{Q} .
- c) If X is a local martingale w.r.t. \mathbb{Q} then $X - \int Z^{-1} d[X, Z]$ is a local mart. w.r.t. \mathbb{P} .
- d) X is a semimartingale w.r.t. \mathbb{P} if and only if it is a semimartingale w.r.t. \mathbb{Q} .

Would you like to put your integration skills to the test on May 8, 2026? Then get ready for this year's *Bonn Integration Bee!* Great prizes await you. The event starts at 4:00 p.m. sharp. For more information, visit our website: <https://bibee.fsmath-bonn.de>.



If you don't want to participate yourself but are still interested in the problems, you're also welcome to come by as a spectator—it's going to be an exciting evening! The student council will be selling cold drinks, and there will be free food.

P.S.: You can find the problems from last time, including the solutions, on our website!