Institut für Angewandte Mathematik
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“Stochastic Analysis”, Problem Sheet 4

Please hand in your solutions by Wednesday, 13 May, 12:00.

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1. (Passage time to a sloping line). Let $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{A}, \mathbb{P})$ be a one-dimensional Brownian motion with $X_0 = 0$, and let $a > 0$. Recall that by the reflection principle, the law of the first passage time $T_a = \inf\{t \geq 0 : X_t = a\}$ is absolutely continuous with density

$$f_{T_a}(t) \;=\; at^{-3/2}\,\varphi(a/\sqrt{t})\,\mathbf{1}_{(0,\infty)}(t).$$

Here φ denotes the standard normal density.

a) For $b \in \mathbb{R}$ let $T_L = \inf\{t \geq 0 : X_t = a + bt\}$ denote the first passage time to the line $y = a + bt$. Show that

$$\mathbb{P}[T_L \leq t] \;=\; \mathbb{E}\left[e^{-bX_t - b^2 t/2}; T_a \leq t\right] \;=\; \int_0^t e^{-ab-b^2 s/2}\, as^{-3/2}\varphi\left(a/\sqrt{s}\right) \mathrm{d}s.$$

Conclude that the law of T_L is absolutely continuous with density

$$f_{T_L}(t) \;=\; at^{-3/2}\,\varphi\left((a+bt)/\sqrt{t}\right)\,\mathbf{1}_{(0,\infty)}(t).$$

b) Show that for any fixed $b > 0$,

$$\mathbb{E}\left[e^{-bX_t}\max_{s\leq t} X_s\right] \;\sim\; \frac{1}{2b}\,e^{b^2 t/2} \qquad \text{asymptotically as } t \to \infty.$$

2. (Brownian motion writes your name). Prove that Brownian motion in $\mathbb{R}^2$ will write your name (in cursive script, without dotted i’s or crossed t’s).
To get the pen rolling, first take $(B_t)_{t\in[0,1]}$ on $(\Omega, \mathcal{A}, \mathbb{P})$ to be a two-dimensional Brownian motion, and note that for any $[a,b] \subseteq [0,1]$ the process

$$X_t^{a,b} = (b-a)^{-1/2}(B_{a+t(b-a)} - B_a)$$

is again a Brownian motion on $[0,1]$. Now, take $g\colon [0,1] \to \mathbb{R}^2$ to be a parametrisation of your name, and note that Brownian motion writes your name (to precision ε) on the interval $[a,b]$ if

$$\sup_{0\leq t\leq 1} |X_t^{a,b} - g(t)| \leq \varepsilon\,. \tag{1}$$

a) Let A_k denote the event that inequality (1) holds for $a = 2^{-k-1}$ and $b = 2^{-k}$. Show that if $\mathbb{P}[A_1] > 0$ then infinitely many of the A_k occur with probability one.

b) Show that

$$\mathbb{P}\Big[\sup_{0\le t\le 1} |B_t| \le \varepsilon\Big] > 0\,.$$

c) Finally, prove that

$$\mathbb{P}\Big[\sup_{0\le t\le 1} |B_t - g(t)| \le \varepsilon\Big] > 0\,,$$

and complete the solution of the problem.

3. (Change of measure for continuous semimartingales). Let $(\mathcal{F}_t)$ be a filtration on $(\Omega, \mathcal{A})$, and let $\mathbb{P}$ and $\mathbb{Q}$ be probability measures that are mutually absolutely continuous on $\mathcal{F}_t$ for any $t \in [0, \infty)$ with densities $Z_t = \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\Big|_{\mathcal{F}_t}$. We assume that (Z_t) is a *continuous* martingale. Show that the following statements hold for an adapted continuous process (X_t):

a) X is a martingale w.r.t. $\mathbb{P}$ if and only if $X \cdot Z$ is a martingale w.r.t. $\mathbb{Q}$.

b) X is a local martingale w.r.t. $\mathbb{P}$ if and only if $X \cdot Z$ is a local martingale w.r.t. $\mathbb{Q}$.

c) If X is a local martingale w.r.t. $\mathbb{Q}$ then $X - \int Z^{-1}\,\mathrm{d}[X, Z]$ is a local mart. w.r.t. $\mathbb{P}$.

d) X is a semimartingale w.r.t. $\mathbb{P}$ if and only if it is a semimartingale w.r.t. $\mathbb{Q}$.