Institut für Angewandte Mathematik
Summer semester 2026
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"Stochastic Analysis", Problem Sheet 3

Please hand in your solutions by Wednesday, 6 May, 12:00.

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1. (Weak solutions and the martingale problem). Consider a solution of an SDE of the form

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t,$$

driven by a d-dimensional Brownian motion B and with locally bounded coefficients $b\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$. The corresponding infinitesimal generator is given by

$$\mathcal{L} \;=\; \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}\frac{\partial^2}{\partial x^i \partial x^j} \;+\; \sum_{i=1}^{d} b^i \frac{\partial}{\partial x^i} \qquad \text{where } a = \sigma\sigma^\top.$$

a) Show that the following conditions are all equivalent:

(i) For any $f \in C^2(\mathbb{R}^d)$, the process $M_t^f = f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)\,\mathrm{d}s$ is a continuous local martingale.

(ii) For any $v \in \mathbb{R}^d$, the process $M_t^v = v\cdot\left(X_t - X_0 - \int_0^t b(X_s)\,\mathrm{d}s\right)$ is a continuous local martingale with quadratic variation

$$[M^v]_t = \int_0^t v\cdot a(X_s)v\,\mathrm{d}s.$$

(iii) For any $v \in \mathbb{R}^d$, the process

$$Z_t^v := \exp\left(v\cdot\left(X_t - X_0 - \int_0^t b(X_s)\,\mathrm{d}s\right) - \frac{1}{2}\int_0^t v\cdot a(X_s)v\,\mathrm{d}s\right).$$

is a continuous local martingale.

Hint: In order to prove that (iii) implies (i) it is enough to consider functions of the form $f(y) = \exp(v\cdot y)$. The general case follows since linear combinations of exponentials are dense in C^2 w.r.t. uniform convergence on compact sets of the functions and their first two derivatives (this may be assumed without proof).

b) Further, show that these conditions imply that

$$(f(X_t)/f(X_0))\exp\left(-\int_0^t (\mathcal{L}f/f)(X_s)\,\mathrm{d}s\right)$$

is a local martingale for any strictly positive C^2 function f.

2. (Brownian motion on the unit sphere). Let $Y_t = B_t/|B_t|$ where $(B_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^d$, $d > 2$. Prove that the time-changed process

$$Z_a = Y_{T_a}, \qquad T = A^{-1} \text{ with } A_t = \int_0^t |B_s|^{-2}\,\mathrm{d}s \ ,$$

is a diffusion taking values in the unit sphere $S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ with generator

$$\mathcal{L}f(x) \;=\; \frac{1}{2}\left(\Delta f(x) - \sum_{i,j} x^i x^j \frac{\partial^2 f}{\partial x^i \partial x^j}(x)\right) - \frac{d-1}{2}\sum_i x^i \frac{\partial f}{\partial x^i}(x), \qquad x \in S^{d-1}.$$

3. (Martingales as time-changed Brownian motions). Let $(\mathcal{F}_t)$ be a right-continuous filtration on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and suppose that (M_t) is a continuous local $(\mathcal{F}_t)$ martingale.

a) Let S and T be $(\mathcal{F}_t)$ stopping times such that $S \leq T$. Prove that if $[M]_S = [M]_T < \infty$ almost surely, then M is almost surely constant on the stochastic interval $[S, T]$.

b) Use this fact to complete the missing steps in the proof of Theorem 1.7 in the lecture notes.