Institut für Angewandte Mathematik
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Andreas Eberle, Francis Lörler

"Stochastic Analysis", Problem Sheet 2

Please hand in your solutions by Wednesday, 29 April, 12:00.

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1. (The jumps of a càdlàg function).

a) Prove that if I is a compact interval, then for any càdlàg function $x\colon I \to \mathbb{R}$, the set $\{s \in I : |\Delta x_s| > \varepsilon\}$ is finite for any $\varepsilon > 0$. Here $\Delta x_s = x_s - x_{s-}$ denotes the size of the jump at s.

b) Conclude that any càdlàg function $x\colon [0,\infty) \to \mathbb{R}$ has at most countably many jumps.

c) Show that a uniform limit of a sequence of càdlàg functions is again càdlàg.

2. (The jumps of a Lévy process). Let $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{A}, \mathbb{P})$ be a Lévy process with state space $\mathbb{R}^d$, and let

$$\mathcal{M}_c^+ = \left\{ \sum \delta_{y_i} : (y_i) \text{ finite or countable sequence in } \mathbb{R}^d \setminus \{0\} \right\}$$

denote the set of all counting measures on $\mathbb{R}^d \setminus \{0\}$. The measure-valued process $N_t\colon \Omega \to \mathcal{M}_c^+$ defined by

$$N_t(B) \;=\; \sum_{\substack{s\leq t\\ \Delta X_s \neq 0}} \delta_{\Delta X_s}(B)\,, \qquad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})\,,$$

encodes the jumps of the Lévy process.

a) Show that $(N_t(B))_{t\geq 0}$ is a Poisson process for any $B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$.
Hint: You may use the strong Markov property for Lévy processes without a proof.

b) Conclude that there exists a unique σ-finite measure ν on $\mathcal{B}(\mathbb{R}^d \setminus \{0\})$ such that

$$\mathbb{E}[N_t(B)] \;=\; t\nu(B) \qquad \text{for all } t \geq 0 \text{ and } B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})\,. \tag{1}$$

The measure ν is called the *jump intensity measure* of the Lévy process (X_t).

c) Show that $\nu(|y| > \varepsilon) < \infty$ for all $\varepsilon > 0$. *Hint: Exercise 1.*

3. (Characterisation of stable processes). Let $\alpha \in (0,2]$ and suppose that $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{A}, \mathbb{P})$ is a real-valued Lévy process with $X_0 = 0$.

a) Show that if $(X_t)_{t\geq 0}$ is strictly α-stable, then its jump intensity measure ν defined by (1) satisfies

$$\nu(cB) \;=\; c^{-\alpha}\nu(B) \qquad \text{for all } B \in \mathcal{B}(\mathbb{R}\setminus\{0\}) \text{ and } c > 0.$$

b) Show that the following statements are equivalent:

(i) $(X_t)_{t\geq 0}$ is strictly α-stable.

(ii) $\psi(cp) = c^\alpha \psi(p)$ for any $c \geq 0$ and $p \in \mathbb{R}$.

(iii) There exist constants $\sigma \geq 0$ and $\mu \in \mathbb{R}$ such that

$$\psi(p) = |p|^\alpha(\sigma^\alpha + i\mu\,\mathrm{sgn}(p))\,.$$

4. (Martingales of compound Poisson processes). Consider a compound Poisson process given by

$$X_t \;=\; \sum_{i=1}^{N_t} Y_i, \qquad t \geq 0,$$

with a Poisson process (N_t) of intensity $\lambda > 0$ and independent i.i.d. random variables Y_i, $i \in \mathbb{N}$, with distribution μ, expectation m and finite variance σ^2.

a) Prove that (X_t) is a Lévy process with characteristic exponent

$$\psi(p) \;= \lambda \int (1 - \exp(ip\cdot y))\,\mu(\mathrm{d}y)\,.$$

b) Show that $M_t = X_t - m\lambda t$ is a martingale.

c) Suppose that μ is a normal distribution. For which values of λ is the process

$$Z_t \;=\; \exp\left(-X_t + \left(m - \tfrac{1}{2}\sigma^2\right)t\right)$$

a supermartingale? Consider the cases $m = \sigma^2/2$, $m < \sigma^2/2$ and $m > \sigma^2/2$.