

## “Stochastic Analysis”, Problem Sheet 1

Please hand in your solutions by Wednesday, April 22, 12:00.

**1. (Ornstein-Uhlenbeck process).** An Ornstein-Uhlenbeck process on  $\mathbb{R}$  is a solution to the SDE

$$dX_t = -X_t dt + \sqrt{2} dB_t,$$

where  $(B_t)_{t \geq 0}$  is a one-dimensional Brownian motion.

a) Let  $(W_t)_{t \geq 0}$  be a one-dimensional Brownian motion with  $W_0 = 0$ . Show that  $e^{-t}W_{e^{2t}}$  is an Ornstein-Uhlenbeck process.

b) Let  $(X_t)_{t \geq 0}$  be an Ornstein-Uhlenbeck process with  $X_0 = 0$  and let

$$Y_t = \int_0^t X_s ds.$$

Show that  $(Y_t)_{t \geq 0}$  is a mean-zero Gaussian process and calculate the covariance function.

**2. (Integrated Brownian motion).** Let  $(B_t)_{t \geq 0}$  be a standard Brownian motion on  $\mathbb{R}^d$ , and let

$$Y_t = \int_0^t B_s ds.$$

a) Calculate the mean and covariance function of  $(Y_t)_{t \geq 0}$ .

b) Show that the joint process  $(Y_t, B_t)_{t \geq 0}$  is Markovian and determine its generator.

**3. (Smooth densities I).** Let  $(B_t)_{t \geq 0}$  be a Brownian motion in  $\mathbb{R}^3$  and consider the Stratonovich and Itô stochastic differential equations

$$\begin{aligned} \circ dX_t &= X_t \times \circ dB_t, \\ dX_t &= X_t \times dB_t. \end{aligned}$$

Do the solutions of these equations have a smooth density?

**4. (Smooth densities II).** Consider the stochastic differential equation

$$\begin{aligned} dX_t &= V_t \\ dV_t &= b(X_t) dt - \gamma(X_t)V_t dt + dB_t, \end{aligned}$$

where  $(B_t)_{t \geq 0}$  is a Brownian motion on  $\mathbb{R}^d$  and  $b: \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $\gamma: \mathbb{R}^d \rightarrow \mathbb{R}$  are smooth. Find conditions on  $b$  and  $\gamma$  under which the process  $X_t$  has a smooth density.