Institut für angewandte Mathematik
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„Stochastic Analysis", Problem Sheet 9

Please hand in your solutions before 12 noon on Wednesday June 19 into the marked post box opposite to the maths library.

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1. (SDE with linear coefficients). Consider the SDE

$$dX_t \;=\; A_t X_t\,dt \;+\; \sum_{k=1}^{d} \sigma_t^k X_t\,dB_t^k, \tag{1}$$

where (B,P) is a d-dimensional Brownian motion, and $A, \sigma^1, \dots, \sigma^d$ are $(\mathcal{F}_t^{B,P})$ adapted, bounded, $(n \times n)$ matrix-valued continuous processes.

a) Show that for a given initial value $a \in \mathbb{R}^n$, the equation has a unique strong solution.

b) Determine the solution in the case $n = 1$ explicitly.

c) Suppose that $X = (X_t)$ is a fundamental solution of (1), i.e., X is an $(n \times n)$ matrix-valued process that satisfies (1) with initial condition $X_0 = I_n$. Show that almost surely, X_t is an invertible matrix for every t, and the inverse process $Z_t = X_t^{-1}$ satisfies

$$dZ_t \;=\; Z_t\left(\sum_{k=1}^{d}\left(\sigma_t^k\right)^2 - A_t\right)dt \;-\; \sum_{k=1}^{d} Z_t \sigma_t^k\,dB_t^k, \qquad Z_0 \;=\; I_n. \tag{2}$$

Hint: Define Z as the solution of (2), and verify that $Z = X^{-1}$ almost surely.

2. (Burkholder-Davis-Gundy revisited). This exercise contains an alternative proof of the upper bound in the BDG inequality. Let $p > 0$.

a) Suppose X and Y are non-negative random variables such that, for some $\beta > 1$, $\delta \in (0,1)$ and $\varepsilon \in (0, \beta^{-p}/2)$, we have

$$\mathbb{P}[X > \beta\lambda, Y < \delta\lambda] \;\le\; \varepsilon\,\mathbb{P}[X \ge \lambda] \qquad \text{for all } \lambda > 0.$$

Show that there exists a finite constant C_p (depending only on p, β, δ and ε but not on X and Y) such that

$$\mathbb{E}[X^p] \;\le\; C_p\,\mathbb{E}[Y^p].$$

b) Let $(B, \mathbb{P})$ be a Brownian motion starting at 0. Use part (i) to show Burkholder's inequality for B. That is, show that there exists a finite constant C_p such that for all finite stopping times T,

$$\mathbb{E}\left[(B_T^*)^p\right] \;\leq\; C_p \,\mathbb{E}[T^{p/2}].$$

Hint. Let $U = \inf\{t \geq 0 : |B_t| > \lambda\}$. *Show first that*

$$\mathbb{P}[B_T^* > \beta\lambda, T^{1/2} < \delta\lambda] \leq \mathbb{P}\left[\sup_{U \leq t \leq U + \delta^2\lambda^2} |B_t - B_U| > (\beta - 1)\lambda,\, U < \infty\right].$$

c) Now conclude from part b) that the upper bound in the BDG inequality holds for an arbitrary continuous local martingale $(M, \mathbb{P})$ with $M_0 = 0$.

3. (Itô calculus for Lévy processes III). We consider a real-valued Lévy martingale

$$X_t \;=\; \int h(y)\; \tilde{N}_t(dy),$$

where $(\tilde{N}_t)$ is a compensated Poisson point process on a σ-finite measure space $(S, \mathcal{B}, \nu)$, and $h \in \mathcal{L}^2(\nu)$. The goal of this exercise is to compute the exponential of X, i.e., the solution Z to the equation

$$dZ_t \;=\; Z_{t-}\, dX_t, \qquad Z_0 = 1. \tag{3}$$

To this end, we try the ansatz

$$Z_t = \exp(L_t), \qquad L_t \;=\; \int g(y)\; \tilde{N}_t(dy) + bt,$$

where b is a real constant, and g is a function in $\mathcal{L}^2(\nu)$.

a) Show that for $f \in C^2(\mathbb{R})$,

$$f(L_t) - f(L_0) \;=\; \int_{(0,t]\times S} \{f(L_{s-} + g(y)) - f(L_{s-})\}\; \tilde{N}(ds\, dy) \;+\; \int_0^t (\mathcal{L}f)(L_s)\, ds,$$

where

$$(\mathcal{L}f)(x) \;=\; bf'(x) + \int \{f(x + g(y)) - f(x) - f'(x)g(y)\}\, \nu(dy).$$

Hint: Note that $\int g(y)\, \tilde{N}_t(dy) = \int z\, (\tilde{N}_t \circ g^{-1})(dz)$.

b) Determine b such that Z is a local martingale, and show that in this case,

$$Z_t \;=\; 1 + \int_{(0,t]\times S} Z_{s-}(e^{g(y)} - 1)\, \tilde{N}(ds\, dy).$$

c) Now determine a solution of Equation (3).