Institut für angewandte Mathematik
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„Stochastic Analysis", Problem Sheet 8

Please hand in your solutions before 12 noon on Wednesday June 5 into the marked post box opposite to the maths library.

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1. (Brownian local time). Let L^y denote the local time in y of a one-dimensional Brownian motion $(B, \mathbb{P}_x)$ starting at $x \in \mathbb{R}$.

a) Let $T = \inf\{t \geq 0 : B_t \notin (a, b)\}$ with $a \leq x \leq y \leq b$. Show that

$$\mathbb{E}_x[L_T^y] \;=\; \frac{2(x-a)(b-y)}{b-a}.$$

b) Show that for all $a > 0$, the processes $(L_{at}^0)_{t\geq 0}$ and $(\sqrt{a}L_t^0)_{t\geq 0}$ have the same law under $\mathbb{P}_0$.

2. (Some applications of Burkholder's inequality). Let $(W, \mathbb{P})$ be a one-dimensional Brownian motion starting at 0.

a) Suppose that X is a solution to an SDE

$$dX_t \;=\; \sigma(X_t)\, dW_t \;+\; b(X_t)\, dt, \qquad X_0 = x_0,$$

where $x_0 \in \mathbb{R}$, and σ and b are continuous real-valued functions such that

$$|\sigma(x)| + |b(x)| \;\leq\; c\,(1 + |x|) \qquad \text{for some } c \in (0, \infty).$$

Show that for every $t_0 \in [0, \infty)$ and $p \in [2, \infty)$, there is a finite constant C such that

$$\mathbb{E}\left[\sup_{s\leq t_0} |X_s|^p\right] \;\leq\; C\,\left(1 + |x_0|^p\right).$$

b) Show that $\mathbb{E}[W_T] = 0$ holds for any stopping time T such that $\mathbb{E}[T^{1/2}] < \infty$.

3. (Itô calculus for Lévy processes II). In continuation of Exercise 6.1, we now extend Itô's formula to a real-valued Lévy process with jump intensity measure ν satisfying only

$$\int (|y| \wedge |y|^2)\, \nu(dy) \;<\; \infty.$$

Let

$$X_t \;=\; \sigma B_t \,+\, bt \,+\, \int y\ \tilde{N}_t(dy),$$

where σ and b are real constants, (B_t) is a Brownian motion, (N_t) is an independent Poisson point process on $\mathbb{R} \setminus \{0\}$ with intensity measure ν and corresponding Poisson random measure $N(dt\,dy)$, and $\tilde{N}_t = N_t - t\nu$.

a) Let $f \in C_b^2(\mathbb{R})$. Proceeding similarly as in Exercise 6.1, show that almost surely,

$$\begin{aligned} f(X_t) \;&=\; f(X_0) \,+\, \int_0^t (\sigma f')(X_s)\, dB_s \,+\, \int_{(0,t]\times\mathbb{R}} f'(X_{s-})y\, \tilde{N}(ds\,dy) \\ &\quad + \int_0^t \left(\frac{1}{2}\sigma^2 f'' + bf'\right)(X_s)\, ds \\ &\quad + \int_{(0,t]\times\mathbb{R}} \left(f(X_{s-}+y) - f(X_{s-}) - f'(X_{s-})y\right)\, N(ds\,dy). \end{aligned}$$

Why is it not possible to write Itô's formula in a similar way as in Exercise 6.1 ?

b) Conclude that (X_t) solves the martingale problem for the operator

$$(\mathcal{L}f)(x) \;=\; \frac{1}{2}(\sigma^2 f'')(x) \,+\, (bf')(x) \,+\, \int \left(f(x+y) - f(x) - f'(x)y\right)\, \nu(dy)$$

with domain $\mathrm{Dom}(\mathcal{L}) = C_b^2(\mathbb{R})$.