Institut für angewandte Mathematik
Sommersemester 2018/19
Andreas Eberle, Kaveh Bashiri

„Stochastic Analysis", Problem Sheet 7

Please hand in your solutions before 12 noon on Wednesday May 29 into the marked post box opposite to the maths library.

---

1. (Burkholder's inequality). Prove that for a given $p \in [4, \infty)$, there exists a global constant $c_p \in (1, \infty)$ such that

$$\mathbb{E}\left[[M]_\infty^{p/2}\right] \;\leq\; c_p \, \mathbb{E}\left[\sup_{t \geq 0} |M_t|^p\right]$$

holds for any continuous local martingale $(M_t)_{t \in [0,\infty)}$ with $M_0 = 0$.

Hint: Start from the identity $M_t^2 = 2 \int_0^t M \, dM \; + \; [M]_t$.

2. (Brownian local time). Let L^a be the local time in $a \in \mathbb{R}$ of a one-dimensional Brownian motion $(B_t)_{t \geq 0}$ starting at 0.

a) Show that

$$L_t^a \;=\; \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{(a-\varepsilon, a+\varepsilon)}(B_s) \, ds.$$

b) Prove the **Skorohod Lemma**: If $y : [0, \infty) \to \mathbb{R}$ is a real-valued continuous function with $y(0) = 0$ then there exists a unique pair (x, k) of functions on $[0, \infty)$ such that

(i) $x = y + k$,

(ii) x is non-negative,

(iii) k is non-decreasing, continuous, vanishing at zero, and the measure dk_t is carried by the set $\{t : \, x(t) = 0\}$.

Moreover, the function k is given by $k(t) = \sup_{s \leq t}(-y(s))$.

c) Conclude that L_t^0 and $S_t := \sup_{s \leq t} B_s$ have the same law.

3. (Dominated Convergence Theorem for stochastic integrals). Let X be a continuous semimartingale on a filtered probability space $(\Omega, \mathcal{A}, P, (\mathcal{F}_t))$, and let (G^n) be a sequence of predictable processes such that almost surely, for any $t \geq 0$,

$$G_t^n \;\longrightarrow\; G_t \qquad \text{as } n \to \infty.$$

Show that if there exists a finite constant $C \in \mathbb{R}_+$ such that $|G_t^n| \leq C$ for any t and n, then

$$\int G^n \, dX \;\longrightarrow\; \int G \, dX$$

uniformly on compact time-intervals in probability.