**Institut für angewandte Mathematik**
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# „Stochastic Analysis", Problem Sheet 6

**Please hand in your solutions before 12 noon on Wednesday May 22 into the marked post box opposite to the maths library.**

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**1. (Itô calculus for Lévy processes I).** We consider a real-valued Lévy process

$$X_t \;=\; \sigma B_t \,+\, bt \,+\, \int y\; N_t(dy)$$

Here $\sigma$ and $b$ are real constants, $(B_t)$ is a Brownian motion, and $(N_t)$ is an independent Poisson point process on $\mathbb{R}\setminus\{0\}$ with intensity measure $\nu$ and corresponding Poisson random measure $N(dt\,dy)$ .

a) Suppose first that $\nu$ is a finite measure, and let $f \in C_b^2(\mathbb{R})$. Show that almost surely,

$$\begin{aligned} f(X_t) \;=\; & f(X_0) + \int_0^t \sigma f'(X_s)\,dB_s + \int_0^t \left(\frac{1}{2}\sigma^2 f'' + bf'\right)(X_s)\,ds \\ & + \int_{(0,t]\times\mathbb{R}} \left(f(X_{s-}+y) - f(X_{s-})\right)\, N(ds\,dy). \end{aligned} \tag{1}$$

*Hint: First consider the increments $f(X_{T_n}) - f(X_{T_{n-1}})$ where $0 = T_0 < T_1 < T_2 < \cdots$ are the jump times of the Poisson point process.*

b) Now consider a general intensity measure such that $\int (1 \wedge |y|)\,\nu(dy) < \infty$. Show that as $\epsilon \downarrow 0$, the processes

$$X_t^\epsilon \;:=\; \sigma B_t \,+\, bt \,+\, \int_{|y|\geq\epsilon} y\; N_t(dy)$$

converge to $(X_t)$ in the ucp sense (uniformly on finite time intervals, in probability). Hence conclude that Itô's formula (1) holds true for $(X_t)$.

c) Conclude that $(X_t)$ solves the martingale problem for the operator

$$(\mathcal{L}f)(x) \;=\; \frac{1}{2}\sigma^2 f''(x) \,+\, bf'(x) \,+\, \int \left(f(x+y) - f(x)\right)\,\nu(dy), \quad \mathrm{Dom}(\mathcal{L}) = C_b^2(\mathbb{R}).$$

**2. (Stratonovich to Itô conversion).**

a) Show that the "associative law"

$$\int X \circ d\left(\int Y \circ dZ\right) \;=\; \int XY \circ dZ$$

holds for Stratonovich integrals of continuous semimartingales $X, Y, Z$.

b) We consider a Stratonovich SDE in $\mathbb{R}^n$ of the form

$$\circ dX_t \quad = \quad b(X_t)\ dt + \sum_{k=1}^{d} \sigma_k(X_t) \circ dB_t^k, \qquad X_0 \ = \ x_0, \tag{2}$$

with $x_0 \in \mathbb{R}^n$, vector fields $b \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $\sigma_1, \ldots, \sigma_d \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, and an $\mathbb{R}^d$-valued Brownian motion $(B_t)$. Prove that (2) is equivalent to the Itô SDE

$$dX_t \quad = \quad \tilde{b}(X_t)\ dt + \sum_{k=1}^{d} \sigma_k(X_t)\ dB_t^k, \qquad X_0 \ = \ x_0, \tag{3}$$

where $\tilde{b} \ := \ b + \frac{1}{2}\sum_{k=1}^{d} \sigma_k \cdot \nabla \sigma_k$. Conclude that if $\tilde{b}$ and $\sigma_1, \ldots, \sigma_d$ are Lipschitz continuous, then there is a unique strong solution of (2).

**3. (Brownian motion on hypersurfaces).** Let $f \in C^\infty(\mathbb{R}^{n+1})$ such that $f(x) \to \infty$ as $|x| \to \infty$, and suppose that $c \in \mathbb{R}$ is a regular value of $f$, i.e., $\nabla f(x) \neq 0$ for any $x \in f^{-1}(c)$. Then by the implicit function theorem, the level set

$$M_c \quad = \quad f^{-1}(c) \quad = \quad \{x \in \mathbb{R}^{n+1} \ : \ f(x) = c\}$$

is a smooth compact $n$-dimensional submanifold of $\mathbb{R}^{n+1}$. For $x \in M_c$, the vector

$$\mathfrak{n}(x) \quad = \quad \nabla f(x)/|\nabla f(x)| \quad \in S^n$$

is the **unit normal** to $M_c$ at $x$. The **tangent space** to $M_c$ at $x$ is the orthogonal complement

$$T_x M_c \quad = \quad \operatorname{span}\{\mathfrak{n}(x)\}^\perp.$$

Let $P(x) : \mathbb{R}^{n+1} \to T_x M_c$ denote the orthogonal projection onto the tangent space w.r.t. the Euclidean metric, i.e.,

$$P(x)v \quad = \quad v - v \cdot \mathfrak{n}(x)\ \mathfrak{n}(x), \quad v \in \mathbb{R}^{n+1}.$$

For $k \in \{1, \ldots, n+1\}$, we set $P_k(x) = P(x)e_k$. A **Brownian motion on the hypersurface $\boldsymbol{M_c}$** with initial value $x_0 \in M_c$ is a solution $(X_t)$ of the Stratonovich SDE

$$\circ dX_t \quad = \quad P(X_t) \circ dB_t \quad = \quad \sum_{k=1}^{n+1} P_k(X_t) \circ dB_t^k, \qquad X_0 \ = \ x_0, \tag{4}$$

with respect to a Brownian motion $(B_t)$ on $\mathbb{R}^{n+1}$.

a) Prove that almost surely, there exists a unique strong solution $(X_t)_{t \in [0,\infty)}$ of (4), and $(X_t)$ stays on the submanifold $M_c$ for all times.

b) Prove that the SDE (4) can be written in Itô form as

$$dX_t \quad = \quad P(X_t)\ dB_t - \frac{n}{2}\,\kappa(X_t)\,\mathfrak{n}(X_t)\ dt$$

where $\kappa(x) = \frac{1}{n} \operatorname{div} \mathfrak{n}(x)$ is the mean curvature of $M_c$ at $x$.

*Hint: Note that* $\mathfrak{n} \cdot \nabla \mathfrak{n} = 0$ *and* $\mathfrak{n} \cdot \nabla P_k + P_k \cdot \nabla \mathfrak{n} = 0$. *Why do these identities hold ?*