Institut für angewandte Mathematik
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„Stochastic Analysis", Problem Sheet 5

Please hand in your solutions before 12 noon on Wednesday May 15 into the marked post box opposite to the maths library.

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1. (Predictable, optional and progressive σ-algebra). Let $(\mathcal{F}_t)$ be a filtration on a set Ω. Recall that the *predictable σ-algebra* $\mathcal{P}$ on $\Omega \times [0, \infty)$ is generated by the sets $A \times (s,t]$ with $0 \leq s < t$ and $A \in \mathcal{F}_s$.

a) Show that

$$\mathcal{P} \;=\; \sigma(\mathcal{L}) \;=\; \sigma(\mathcal{C})$$

where $\mathcal{L}$ and $\mathcal{C}$ denote the spaces consisting of all $(\mathcal{F}_t)$ adapted left-continuous resp. continuous processes $(\omega, t) \mapsto X_t(\omega)$.

b) The *optional σ-algebra* $\mathcal{O} = \sigma(\mathcal{D})$ and the *progressive σ-algebra* $\mathcal{A} = \sigma(\Pi)$ are generated by the spaces $\mathcal{D}$ and Π consisting of all $(\mathcal{F}_t)$ adapted càdlàg processes, respectively all $(\mathcal{F}_t)$ progressively measurable processes. Show that

$$\mathcal{P} \;\subseteq\; \mathcal{O} \;\subseteq\; \mathcal{A}.$$

c) Show that if T is an $(\mathcal{F}_t)$ stopping time then the set

$$[0,T] \;:=\; \{(\omega,t) \in \Omega \times [0,\infty) : t \leq T(\omega)\}$$

is predictable, and

$$[0,T) \;:=\; \{(\omega,t) \in \Omega \times [0,\infty) : t < T(\omega)\}$$

is optional. Furthermore, show that if T is a predictable stopping time, then $[0,T)$ is predictable as well.

2. (Jumps and structure of Lévy processes).

a) Show that the probability that a Lévy process (X_t) jumps at a given fixed time t is zero.

b) Suppose (X_t) is a Lévy martingale without Brownian component (i.e. $\sigma = 0$ in the Lévy-Ito decomposition). Show that if the total jump intensity $\lambda = \nu(\mathbb{R})$ is finite, then X is a compensated compound Poisson process with jump intensity measure ν.

c) Conclude that any Lévy martingale without Brownian component is a limit of compensated compound Poisson processes.

3. (Quadratic variation of Lévy processes). Let N be a Poisson point process on $\mathbb{R}\setminus\{0\}$ with σ-finite intensity measure ν satisfying $\int |y|^2\,\nu(dy) < \infty$, and let $\tilde{N}$ be the corresponding compensated Poisson point process. We consider the Lévy martingales

$$X_t \;=\; \int_{\mathbb{R}\setminus\{0\}} y\,\tilde{N}_t(dy) \qquad \text{and} \qquad Y_t \;=\; \sigma\,B_t,$$

where σ is a real constant, and B is Brownian motion that is defined on the same probability space as N.

a) Suppose first that ν is a finite measure. Show that

$$[X]_t \;=\; \sum_{s\le t}(\Delta X_s)^2 \qquad \text{and} \qquad [X,Y]_t \;=\; 0. \tag{1}$$

b) In order to extend the result to infinite intensity measures, prove that for $u\in(0,\infty)$ and càdlàg martingales $M,N\in M_d^2([0,u])$,

$$\mathbb{E}\left[\sup_{t\le u}\,[M,N]_t\right] \le \|M\|_{M_d^2([0,u])}\|N\|_{M_d^2([0,u])}.$$

Here you may assume without proof that the covariation $[M,N]$ exists, $MN-[M,N]$ is a martingale, and for an arbitrary partition sequence (π_n) with $\mathrm{mesh}(\pi_n)\to 0$,

$$[M,N]_t \;=\; \lim_{n\to\infty}\sum_{s\in\pi_n}(M_{s'\wedge t}-M_{s\wedge t})(N_{s'\wedge t}-N_{s\wedge t})$$

holds w.r.t. convergence in probability, uniformly for $t\in[0,u]$.

c) Now conclude that (1) still holds true if the intensity measure ν is not finite.

Remark. Note that we did not assume that X and Y are independent ! Using Itô calculus for jump processes, one can conclude from the vanishing of the covariation that a Lévy jump process without Brownian component and a Lévy diffusion are ***always*** *independent.*