Institut für angewandte Mathematik
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„Stochastic Analysis", Problem Sheet 4

Please hand in your solutions before 12 noon on Wednesday May 8 into the marked post box opposite to the maths library.

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1. (Construction of Poisson point processes). Let $(S, \mathcal{B}, \nu)$ be a σ-finite measure space with total mass $\nu(S) = \lambda$.

a) Suppose first that $\lambda \in (0, \infty)$. Prove that

$$N_t \;=\; \sum_{j=1}^{K_t} \delta_{\eta_j}\,, \qquad t \geq 0,$$

is a Poisson point process with intensity measure ν provided the random variables η_j, $j \in \mathbb{N}$, are independent with distribution $\lambda^{-1}\nu$, and (K_t) is an independent Poisson process of intensity λ.

b) Now consider the case $\lambda = \infty$. Let $\nu^{(k)}$ $(k \in \mathbb{N})$ be a sequence of finite measures on $(S, \mathcal{B})$ with $\nu = \sum \nu^{(k)}$. Prove that if $(N_t^{(k)})_{t\geq 0}$, $k \in \mathbb{N}$, are independent Poisson point processes on $(S, \mathcal{B})$ with intensity measures $\nu^{(k)}$ then

$$\overline{N}_t \;=\; \sum_{k=1}^{\infty} N_t^{(k)}$$

is a Poisson point process with intensity measure ν.

2. (Geometric Poisson processes and change of measure). Let $(N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$ on a filtered probability space $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t))$.

a) Let $\sigma, \alpha \in \mathbb{R}$ with $\sigma > -1$. Give a meaning to the SDE

$$dS_t \;=\; \sigma\, S_{t-}\, dN_t \;+\; \alpha\, S_t\, dt\,, \qquad S_0 = 1,$$

and find a solution by the ansatz $S_t = \exp(aN_t + bt)$.

b) Given σ, for which value of α is (S_t) a martingale ?

c) Now let $\mu > 0$. Verify that

$$Z_t \;=\; (\mu/\lambda)^{N_t}\, e^{(\lambda-\mu)t}$$

is an $(\mathcal{F}_t)$ martingale with $\mathbb{E}[Z_t] = 1$ for all t.

d) We define a new probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}_1)$ by

$$\tilde{\mathbb{P}}[A] \;=\; \int_A Z_1 \, d\mathbb{P} \qquad \text{for any } A \in \mathcal{F}_1.$$

Verify that $\tilde{\mathbb{E}}[X_t] = \mathbb{E}[X_t Z_t]$ for any $\mathcal{F}_t$ measurable random variable X_t and $t \in [0,1]$. Compute the characteristic function of the process $(N_t)_{t\in[0,1]}$ w.r.t. the new measure $\tilde{\mathbb{P}}$. Conclude that under $\tilde{\mathbb{P}}$, (N_t) is a Poisson process with intensity μ.

3. (Martingales of Poisson point processes).

a) Consider a one-dimensional compound Poisson process given by $X_t = \sum_{i=1}^{K_t} Y_i$ with i.i.d. random variables Y_i $(i \in \mathbb{N})$ with distribution μ, and an independent Poisson process (K_t) of intensity $\lambda > 0$. Verify that the following processes are martingales:

(i) $M_t := X_t - \lambda m t$, provided $Y_1 \in \mathcal{L}^1$ with $\mathbb{E}[Y_1] = m$,

(ii) $M_t^2 - \lambda \sigma^2 t$, provided $Y_1 \in \mathcal{L}^2$ with $\mathbb{E}[Y_1^2] = \sigma^2$,

(iii) $\exp(ipX_t + t\psi(p))$ for any $p \in \mathbb{R}$, where $\psi(p) \;= \lambda \int (1 - \exp(ip\cdot y))\, \mu(dy)$.

b) Now suppose that $(N_t)_{t\geq 0}$ is a Poisson point process with a finite intensity measure ν. Show that the following processes are martingales w.r.t. the filtration $(\mathcal{F}_t^N)$:

(i) $\tilde{N}_t(f) \;=\; N_t(f) - t\int f d\nu$ for any $f \in \mathcal{L}^1(\nu)$,

(ii) $\tilde{N}_t(f)\tilde{N}_t(g) - t\int fg\, d\nu$ for any $f, g \in \mathcal{L}^2(\nu)$,

(iii) $\exp\left(ipN_t(f) + t\int (1 - e^{ipf})\, d\nu\right)$ for any measurable $f : S \to \mathbb{R}$ and $p \in \mathbb{R}$.

4. (Simulation of Lévy processes). Write a computer program that simulates and plots a trajectory of a Lévy jump process with finite jump intensity measure.

The student council of mathematics will organize the math party on 9/05 in N8schicht. The presale will be held on Mon 6/05, Tue 7/05 and Wed 8/05 in the mensa Poppelsdorf. Further information can be found at fsmath.uni-bonn.de