Institut für angewandte Mathematik
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„Stochastic Analysis", Problem Sheet 3

Please hand in your solutions before 12 noon on Tuesday April 30 into the marked post box opposite to the maths library.

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1. (Properties of càdlàg functions).

a) Prove that if I is a compact interval, then for any càdlàg function $x : I \to \mathbb{R}$, the set $\{s \in I : |\Delta x_s| > \varepsilon\}$ is finite for any $\varepsilon > 0$. Conclude that any càdlàg function $x : [0, \infty) \to \mathbb{R}$ has at most countably many jumps.

b) Show that a uniform limit of a sequence of càdlàg functions is again càdlàg.

2. (Brownian motion writes your name). Prove that Brownian motion in $\mathbb{R}^2$ will write your name (in cursive script, without dotted i's or crossed t's).
To get the pen rolling, first take B_t to be a two-dimensional Brownian motion on $[0, 1]$, and note that for any $[a, b] \subset [0, 1]$ the process

$$X_t^{(a,b)} = (b - a)^{-1/2}(B_{a+t(b-a)} - B_a)$$

is again a Brownian motion on $[0, 1]$. Now, take $g : [0, 1] \to \mathbb{R}^2$ to be a parametrization of your name, and note that Brownian motion spells your name (to precision ϵ) on the interval $[a, b]$ if

$$\sup_{0 \le t \le 1} |X_t^{a,b} - g(t)| \le \epsilon. \tag{1}$$

a) Let A_k denote the event that inequality (1) holds for $a = 2^{-k-1}$ and $b = 2^{-k}$. Check that the events A_k are independent, and that one has $P[A_k] = P[A_1]$ for all k. Conclude that if $P[A_1] > 0$ then infinitely many of the A_k will occur with probability one.

b) Show that

$$P\left[\sup_{0 \le t \le 1} |B_t| \le \epsilon\right] > 0.$$

c) Finally, prove that

$$P\left[\sup_{0 \le t \le 1} |B_t - g(t)| \le \epsilon\right] > 0,$$

and complete the solution of the problem.

3. (Change of measure for continuous semimartingales). Let $(\mathcal{F}_t)$ be a filtration on $(\Omega, \mathcal{A})$, and let P and Q be probability measures that are mutually absolutely continuous on $\mathcal{F}_t$ for any $t \in [0, \infty)$ with densities $Z_t = \frac{dP}{dQ}\Big|_{\mathcal{F}_t}$. We assume that (Z_t) is a *continuous* martingale. Show that the following statements hold for an adapted continuous process (X_t):

a) X is a martingale w.r.t. P if and only if $X \cdot Z$ is a martingale w.r.t. Q.

b) X is a local martingale w.r.t. P if and only if $X \cdot Z$ is a local martingale w.r.t. Q.

c) If X is a local martingale w.r.t. Q then $X - \int Z^{-1}\, d[X, Z]$ is a local mart. w.r.t. P.

d) X is a semimartingale w.r.t. P if and only if it is a semimartingale w.r.t. Q.

4. (Exit distributions for compound Poisson processes).
Let $(X_t)_{t \geq 0}$ be a compound Poisson process with $X_0 = 0$ and jump intensity measure $\nu = N(m, 1),\ m > 0$.

a) Determine $\lambda \in \mathbb{R}$ such that $\exp(\lambda X_t)$ is a local martingale.

b) Prove that for $a < 0$,

$$P[T_a < \infty] \;=\; \lim_{b \to \infty} P[T_a < T_b] \;\leq\; \exp(2ma),$$

where

$$T_a := \inf\{t \geq 0 : X_t \leq a\} \qquad \text{and} \qquad T_b := \inf\{t \geq 0 : X_t \geq b\}.$$

Why is it not as easy as for Bessel processes (see Sheet 1) to compute the ruin probability $P[T_a < T_b]$ exactly?