Institut für angewandte Mathematik
Sommersemester 2018/19
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„Stochastic Analysis", Problem Sheet 2

Please hand in your solutions before 12 noon on Thursday April 18 into the marked post box opposite to the maths library.

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1. (Brownian motion on the unit sphere). Let $Y_t = B_t/|B_t|$ where $(B_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^n$, $n > 2$. Prove that the time-changed process

$$Z_a = Y_{T_a}, \qquad T = A^{-1} \text{ with } A_t = \int_0^t |B_s|^{-2}\, ds \;,$$

is a diffusion taking values in the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ with generator

$$\mathcal{L}f(x) \;=\; \frac{1}{2}\left(\Delta f(x) - \sum_{i,j} x^i x^j \frac{\partial^2 f}{\partial x^i \partial x^j}(x)\right) - \frac{n-1}{2}\sum_i x^i \frac{\partial f}{\partial x^i}(x), \qquad x \in S^{n-1}.$$

2. (Martingales as time-changed Brownian motions). Let $(\mathcal{F}_t)$ be a right-continuous filtration on a probability space $(\Omega, \mathcal{A}, P)$, and suppose that (M_t) is a continuous local $(\mathcal{F}_t)$ martingale.

a) Let S and T be $(\mathcal{F}_t)$ stopping times such that $S \leq T$. Prove that if $[M]_S = [M]_T < \infty$ almost surely, then M is almost surely constant on the stochastic interval $[S, T]$.

b) Use this fact to complete the missing steps in the proof of Theorem 1.7 in the lecture notes.

3. (Passage time to a sloping line). Let $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{A}, P)$ be a one-dimensional Brownian motion with $X_0 = 0$, and let $a > 0$.

a) Recall that by the reflection principle, the law of the first passage time $T_a = \inf\{t \geq 0 : X_t = a\}$ is absolutely continuous with density

$$f_{T_a}(t) \;=\; at^{-3/2}\, \varphi(a/\sqrt{t})\, 1_{(0,\infty)}(t).$$

Here φ denotes the standard normal density.

b) For $b \in \mathbb{R}$ let $T_L = \inf\{t \geq 0 : X_t = a + bt\}$ denote the first passage time to the line $y = a + bt$. Show that

$$P[T_L \leq t] \;=\; E_P\left[e^{-bX_t - b^2t/2}; T_a \leq t\right] \;=\; \int_0^t e^{-ab-b^2s/2}\, as^{-3/2}\varphi\left(a/\sqrt{s}\right)\, ds.$$

Conclude that the law of T_L is absolutely continuous with density

$$f_{T_L}(t) \;=\; at^{-3/2}\,\varphi\left((a+bt)/\sqrt{t}\right)\, 1_{(0,\infty)}(t).$$

c) Show that for any fixed $b > 0$,

$$E_P\left[e^{-bX_t}\max_{s\leq t} X_s\right] \sim \frac{1}{2b}\, e^{b^2t/2} \qquad \text{asymptotically as } t \to \infty.$$

4. (Strong Markov property for Lévy processes). Let $(X_t)_{t\geq 0}$ be a Lévy process w.r.t. the filtration $(\mathcal{F}_t)_{t\geq 0}$ and let T be a finite stopping time. Show that $Y_t = X_{T+t} - X_T$ is a process that is independent of $\mathcal{F}_T$, and X and Y have the same law.

Hint: Consider the sequence of stopping times defined by

$$T_n(\omega) = \frac{k+1}{2^n} \qquad \text{if} \quad \frac{k}{2^n} \leq T < \frac{k+1}{2^n}.$$

Notice that $T_n \downarrow T$ as $n \to \infty$. In a first step show that for any $t_1 < t_2 < \ldots < t_m$, $m \geq 1$, any bounded continuous function f on $\mathbb{R}^m$, and any $A \in \mathcal{F}_T$ we have

$$E\left[f(X_{T_n+t_1} - X_{T_n}, \ldots, X_{T_n+t_m} - X_{T_n})1_A\right] = E\left[f(X_{t_1}, \ldots, X_{t_m})\right]\, P[A].$$

5. (A characterization of Poisson processes). Let $(X_t)_{t\geq 0}$ be a Lévy process with $X_0 = 0$ a.s. Suppose that the paths of X are piecewise constant, increasing, all jumps of X are of size 1, and X is not identically 0. Prove that X is a Poisson process.

Hint: Apply the Strong Markov property in Problem 3 to the jump times $(T_i)_{i=1,2,\ldots}$ of X to conclude that the random variables $U_i := T_i - T_{i-1}$ are i.i.d. (with $T_0 := 0$). Then, it remains to show that U_1 is an exponential random variable with some parameter $\lambda > 0$.

Because of the public holidays on easter monday, April 22, and on wednesday, May 1, we will not hand out a problem sheet next week. Thus the next problem sheet after this one will be due on Thursday, May 2.