Institut für angewandte Mathematik
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„Stochastic Analysis", Problem Sheet 10

Please hand in your solutions before 12 noon on Wednesday June 26 into the marked post box opposite to the maths library.

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1. (Reflection coupling of Brownian motions). Let $x, y \in \mathbb{R}^d$. We consider the reflection coupling $(X_t, Y_t)_{t\in[0,\infty)}$ of two Brownian motions X and Y with initial values $X_0 = x$ and $Y_0 = y$,

a) Show that
$$Y_t \;=\; R_{x,y}X_t \quad \text{for } t < T, \quad \text{and} \quad Y_t = X_t \quad \text{for } t \geq T,$$
where $R_{x,y}z$ denotes the mirror image of z reflected at the hyperplane $H_{x,y}$ right between x and y, and T is the coupling time.

b) Conclude that
$$\mathbb{P}[T > t] \;=\; 2\,\Phi\left(\frac{|x-y|}{2\sqrt{t}}\right) - 1.$$
Hint: It might be helpful to consider at first the case $d = 1$ and $y = -x$.

c) Prove that reflection coupling is a *maximal coupling* for Brownian motion, i.e.,
$$\mathbb{P}[X_t \neq Y_t] \;=\; \|p_t(x,\cdot) - p_t(y,\cdot)\|_{TV} \qquad \text{for all } t \geq 0,$$
where p_t denotes the transition function of Brownian motion.

2. (Simulation of stochastic differential equations).

a) Simulate trajectories of a general one-dimensional Itô diffusion process and plot the output.

b) Try your simulation on some examples.

c) Now include a Poisson noise with finite jump intensity into your SDE. Again simulate and plot trajectories of solutions.