Institut für angewandte Mathematik
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„Stochastic Analysis", Problem Sheet 1

Please hand in your solutions before 12 noon on Thursday April 11 into the marked post box opposite to the maths library.

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1. (Weak solutions and the martingale problem). Consider a solution of an SDE of the form

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t,$$

driven by a d-dimensional Brownian motion B and with locally bounded coefficients $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$. The corresponding infinitesimal generator is given by

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i \frac{\partial}{\partial x^i} \qquad \text{where } a = \sigma\sigma^T.$$

a) Show that the following conditions are all equivalent:

(i) For any $f \in C^2(\mathbb{R}^d)$, the process $M_t^f = f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)\,ds$ is a continuous local martingale.

(ii) For any $v \in \mathbb{R}^d$, the process $M_t^v = v \cdot \left(X_t - X_0 - \int_0^t b(X_s)\,ds\right)$ is a continuous local martingale with quadratic variation

$$[M^v]_t = \int_0^t v \cdot a(X_s)v\,ds.$$

(iii) For any $v \in \mathbb{R}^d$, the process

$$Z_t^v := \exp\left(v \cdot \left(X_t - X_0 - \int_0^t b(X_s)\,ds\right) - \frac{1}{2}\int_0^t v \cdot a(X_s)v\,ds\right).$$

is a continuous local martingale.

Hint: In order to prove that (iii) implies (i) it is enough to consider functions of the form $f(y) = \exp(v \cdot y)$. The general case follows, since linear combinations of exponentials are dense in C^2 w.r.t. uniform convergence on compact sets of the functions and their first two derivatives (- this may be assumed without proof).

b) Further, show that these conditions imply that

$$(f(X_t)/f(X_0))\exp\left(-\int_0^t (\mathcal{L}f/f)(X_s)\,ds\right)$$

is a local martingale for any strictly positive C^2 function f.

2. (Martingales of compound Poisson processes). Consider a compound Poisson process given by

$$X_t \;=\; \sum_{i=1}^{N_t} Y_i, \qquad t \geq 0,$$

with a Poisson process (N_t) of intensity $\lambda > 0$ and independent i.i.d. random variables Y_i, $i \in \mathbb{N}$, with distribution μ, expectation value m and finite variance σ^2.

a) Prove that (X_t) is a Lévy process with characteristic exponent

$$\psi(p) \;= \lambda \int (1 - \exp(ip \cdot y))\, \mu(dy)\,.$$

b) Show that $M_t := X_t - m\lambda t$ is a martingale.

c) Suppose that μ is a normal distribution. For which values of λ is the process

$$Z_t \;=\; \exp(-X_t + (m - \frac{1}{2}\sigma^2)t)$$

a supermartingale? Consider the cases $m = \sigma^2/2$, $m < \sigma^2/2$ and $m > \sigma^2/2$.

3. (Exit distributions for Bessel processes).

a) Show that the radial process $X_t = |W_t|$ of a d-dimensional Brownian motion is a solution of the SDE

$$dX_t \;=\; \frac{d-1}{2}\frac{1}{X_t}\,dt \,+\, dB_t \tag{1}$$

driven by a one-dimensional Brownian motion (B_t).

b) Now suppose that (X_t) is a solution of (1) for an arbitrary (not necessarily integer valued) constant $d \in \mathbb{R}$.

i) Find a non-constant function $u : \mathbb{R} \to \mathbb{R}$ such that $u(X_t)$ is a local martingale.

ii) Compute the ruin probability $P[T_a < T_b]$ for $0 < a < b$ with $x_0 \in [a, b]$, where

$$T_a := \inf\{t \geq 0 : X_t \leq a\} \qquad \text{and} \qquad T_b := \inf\{t \geq 0 : X_t \geq b\}.$$

iii) Proceeding similarly, determine the mean exit time $E[T]$, where $T = \min\{T_a, T_b\}$.