Institut für angewandte Mathematik
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„Stochastic Analysis", Problem Sheet 0

This exercise will be discussed during the first tutorial.
You can prepare it in advance - submission of the solution is not required.

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1. (Feynman-Kac formula for Itô diffusions). Consider a solution $X_t : \Omega \to \mathbb{R}^n$ of a stochastic differential equation

$$dX_t \;=\; b(t, X_t)\, dt + \sigma(t, X_t)\, dB_t, \qquad X_0 = x,$$

with continuous coefficients driven by an $(\mathcal{F}_t)$ Brownian motion taking values in $\mathbb{R}^d$. Fix $t \in (0, \infty)$, and suppose that $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $V : [0, t] \times \mathbb{R}^n \to [0, \infty)$ are continuous functions. Show that if $u \in C^2((0, t] \times \mathbb{R}^n) \cap C([0, t] \times \mathbb{R}^n)$ is a bounded solution of the equation

$$\begin{aligned} \frac{\partial u}{\partial s}(s, x) &= (\mathcal{L}u)(s, x) - V(s, x)u(s, x) \qquad \text{for } s \in (0, t],\ x \in \mathbb{R}^n, \\ u(0, x) &= \varphi(x), \end{aligned}$$

then u has the stochastic representation

$$u(t, x) \;=\; E_x\left[\varphi(X_t)\, \exp\left(-\int_0^t V(t - s, X_s)\, ds\right)\right].$$

Here $\mathcal{L}$ is the generator of X given by

$$(\mathcal{L}F)(t, x) = \frac{1}{2}\sum_{i,j=1}^n a^{ij}(t, x)\frac{\partial^2 F}{\partial x^i \partial x^j}(t, x) + \sum_{i=1}^n b^i(t, x)\frac{\partial F}{\partial x^i}(t, x) \quad \text{with } a = \sigma\sigma^T.$$

Hint: Consider the time reversal $\hat{u}(s, x) := u(t - s, x)$ of u on $[0, t]$. Show first that $M_r := \exp(-A_r)\hat{u}(r, X_r)$ is a local martingale if $A_r := \int_0^r V(t - s, X_s)\, ds$.

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Course overview

- Transformations and weak solutions of stochastic differential equations
- Lévy processes and Poisson point processes; SDE with jumps
- Extensions of Ito calculus, stochastic flows
- Stability of SDE, variations and coupling for diffusion processes
- Numerical methods for SDE and (Markov Chain) Monte Carlo
- Possibly outlook to Stochastic partial differential equations

Recommended textbooks

The following standard textbooks cover all a broad range of topics in stochastic analysis. Nevertheless, there are substantial differences in style and content between them.

- Rogers, Williams : *Diffusions, Markov processes and martingales, Vol. 2: Ito calculus*, Cambridge UP.
- Bass : *Stochastic Processes,* Springer.
- Applebaum: *Lévy Processes and Stochastic Calculus,* Cambridge UP.
- Protter *: Stochastic integration and differential equations, Springer.*
- Revuz, Yor : *Continuous martingales and Brownian motion*, Springer.
- Le Gall: *Brownian motion, martingales, and stochastic calculus,* Springer.
- Karatzas, Shreve : *Brownian motion and stochastic calculus*, Springer.
- Ikeda, Watanabe: *SDE and diffusion processes*, North Holland.
- Jacod/Shiryaev: *Limit Theorems for Stochastic Processes,* Springer.

Additional references:

- Durrett : *Stochastic calculus*, CRC Press. (Diffusion processes, connections to partial differential equations, approximations)
- Hackenbroch, Thalmaier: *Stochastische Analysis*, Teubner. (Strong solutions, stochastic calculus on manifolds)
- Seppalainen: *Basics of Stochastic Analysis*, http://www.math.wisc.edu/~seppalai/bookpage.html . (Stochastic integration theory for processes with jumps)
- Da Prato: *Introduction to Stochastic Analysis and Malliavin Calculus*, SNS Pisa. (Infinite dimensional analysis, Malliavin calculus)
- Friedman: *Stochastic Differential Equations and Applications*, Dover.
- Liptser/Shiryev: *Statistics of Random Processes I and II,* 2 nd Ed., Springer.
- Shreve: *Stochastic Calculus for Finance II*, Springer.