Institut für angewandte Mathematik
Summer Semester 2015
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"Stochastic Analysis", Problem sheet 9.

Please hand in solutions before Thursday June 25, 15 ct.

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1. (Dominated Convergence Theorem for stochastic integrals) Let X be a continuous semimartingale on a filtered probability space $(\Omega, \mathcal{A}, P, (\mathcal{F}_t))$, and let (G^n) be a sequence of predictable processes such that almost surely, for any $t \geq 0$,

$$G_t^n \longrightarrow G_t \qquad \text{as } n \to \infty.$$

Show that if there exists a finite constant $C \in \mathbb{R}_+$ such that $|G_t^n| \leq C$ for any t and n, then

$$\int G^n \, dX \longrightarrow \int G \, dX$$

uniformly on compact time-intervals in probability.

2. (Brownian local time) Let L^a be the local time in $a \in \mathbb{R}$ of a one-dimensional Brownian motion $(B_t)_{t \geq 0}$ starting at 0.

a) Show that

$$L_t^a = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{(a-\varepsilon, a+\varepsilon)}(B_s) \, ds.$$

b) Prove the **Skorohod Lemma**: If $y : [0, \infty) \to \mathbb{R}$ is a real-valued continuous function with $y(0) = 0$ then there exists a unique pair (x, k) of functions on $[0, \infty)$ such that

(i) $x = y + k$,

(ii) x is non-negative,

(iii) k is non-decreasing, continuous, vanishing at zero, and the measure dk_t is carried by the set $\{t : x(t) = 0\}$.

Moreover, the function k is given by

$$k(t) = \sup_{s \leq t}(-y(s)).$$

c) Conclude that L_t^0 and $S_t := \sup_{s \leq t} B_s$ have the same law.

3. (Stochastic oscillator)

a) Let A and σ be $d \times d$–matrices, and suppose that (B_t) a Brownian motion in $\mathbb{R}^d$. Solve the SDE

$$dZ_t \;=\; AZ_t\,dt \;+\; \sigma\,dB_t\;, \qquad Z_0 = z_0.$$

(First consider $\sigma = 0$, then apply variation of constants)

b) Small displacements from equilibrium (e.g. of a pendulum) with stochastic reset force are described by SDE of type

$$\begin{aligned} dX_t &= V_t\,dt \\ dV_t &= -X_t\,dt \;+\; dB_t \end{aligned}$$

with a one-dimensional Brownian motion B_t. In complex notation:

$$dZ_t \;=\; -iZ_t\,dt + i\,dB_t\,, \qquad \text{where } Z_t \;=\; X_t + iV_t.$$

(i) Solve the SDE with initial conditions $X_0 = x_0$, $V_0 = v_0$.

(ii) Show that X_t is a normally distributed random variable with mean given by the solution of the corresponding deterministic equation.

(iii) Compute the asymptotic variance $\lim_{t\to\infty} \frac{1}{t}\,\mathrm{var}\,(X_t)$.