Institut für angewandte Mathematik
Summer Semester 2015
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"Stochastic Analysis", Problem sheet 8.

Please hand in solutions for Exercise 1 and 2 before Thursday June 11, 15 ct, and solutions for Exercise 3 and 4 before Thursday June 18, 15 ct.

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1. (Change of measure for continuous semimartingales) Let $(\mathcal{F}_t)$ be a filtration on $(\Omega, \mathcal{A})$, and let P and Q be probability measures that are mutually absolutely continuous on $\mathcal{F}_t$ for any $t \in [0, \infty)$ with densities $Z_t = \frac{dP}{dQ}\big|_{\mathcal{F}_t}$. Show that the following statements hold for an adapted continuous process (X_t):

a) X is a martingale w.r.t. P if and only if $X \cdot Z$ is a martingale w.r.t. Q.

b) X is a local martingale w.r.t. P if and only if $X \cdot Z$ is a local martingale w.r.t. Q.

c) If X is a local martingale w.r.t. Q then $X - \int Z^{-1}\, d[X, Z]$ is a local mart. w.r.t. P.

d) X is a semimartingale w.r.t. P if and only if it is a semimartingale w.r.t. Q.

2. (Burkholder's inequality) Prove that for a given $p \in [4, \infty)$, there exists a global constant $c_p \in (1, \infty)$ such that the inequalities

$$c_p^{-1} E\left[[M]_\infty^{p/2}\right] \;\leq\; E\left[\sup_{t\geq 0} |M_t|^p\right] \;\leq\; c_p^{-1} E\left[[M]_\infty^{p/2}\right]$$

hold for any continuous local martingale $(M_t)_{t\in[0,\infty)}$ with $M_0 = 0$.

Hint: First prove the second inequality starting from Itô's formula. For proving the first inequality start from the identity $M_t^2 = 2\int_0^t M\, dM \,+\, [M]_t$.

3. (Stratonovich to Itô conversion)

a) Show that the "associative law"

$$\int X \circ d\left(\int Y \circ dZ\right) \;=\; \int XY \circ dZ$$

holds for Stratonovich integrals of continuous semimartingales X, Y, Z.

b) We consider a Stratonovich SDE in $\mathbb{R}^n$ of the form

$$\circ dX_t \;=\; b(X_t)\, dt + \sum_{k=1}^{d} \sigma_k(X_t) \circ dB_t^k, \qquad X_0 \;=\; x_0, \tag{1}$$

with $x_0 \in \mathbb{R}^n$, vector fields $b \in C(\mathbb{R}^n, \mathbb{R}^n)$, $\sigma_1, \ldots, \sigma_d \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, and an $\mathbb{R}^d$-valued Brownian motion (B_t). Prove that (1) is equivalent to the Itô SDE

$$dX_t \quad = \quad \tilde{b}(X_t)\ dt + \sum_{k=1}^{d} \sigma_k(X_t)\ dB_t^k, \qquad X_0 \ = \ x_0, \tag{2}$$

where $\tilde{b} := b + \frac{1}{2}\sum_{k=1}^{d} \sigma_k \cdot \nabla \sigma_k$.

c) Conclude that if $\tilde{b}$ and $\sigma_1, \ldots, \sigma_d$ are Lipschitz continuous, then there is a unique strong solution of (1).

4. (Brownian motion on hypersurfaces) Let $f \in C^\infty(\mathbb{R}^{n+1})$ and suppose that $c \in \mathbb{R}$ is a regular value of f, i.e., $\nabla f(x) \neq 0$ for any $x \in f^{-1}(c)$. Then by the implicit function theorem, the level set

$$M_c \quad = \quad f^{-1}(c) \quad = \quad \{x \in \mathbb{R}^{n+1} \ : \ f(x) = c\}$$

is a smooth n-dimensional submanifold of $\mathbb{R}^{n+1}$. For example, if $f(x) = |x|^2$ and $c = 1$ then M_c is the n-dimensional unit sphere S^n. For $x \in M_c$, the vector

$$\mathfrak{n}(x) \quad = \quad \frac{\nabla f(x)}{|\nabla f(x)|} \quad \in S^n$$

is the **unit normal** to M_c at x. The **tangent space** to M_c at x is the orthogonal complement

$$T_x M_c \quad = \quad \operatorname{span}\{\mathfrak{n}(x)\}^\perp.$$

Let $P(x) : \mathbb{R}^{n+1} \to T_x M_c$ denote the orthogonal projection onto the tangent space w.r.t. the Euclidean metric, i.e.,

$$P(x)v \quad = \quad v - v \cdot \mathfrak{n}(x)\ \mathfrak{n}(x), \quad v \in \mathbb{R}^{n+1}.$$

For $k \in \{1, \ldots, n+1\}$, we set $P_k(x) = P(x)e_k$. A **Brownian motion on the hypersurface $\boldsymbol{M_c}$** with initial value $x_0 \in M_c$ is a solution (X_t) of the Stratonovich SDE

$$\circ dX_t \quad = \quad P(X_t) \circ dB_t \quad = \quad \sum_{k=1}^{n+1} P_k(X_t) \circ dB_t^k, \qquad X_0 \ = \ x_0, \tag{3}$$

with respect to a Brownian motion (B_t) on $\mathbb{R}^{n+1}$. We now assume for simplicity that M_c is compact. Then, since c is a regular value of f, the vector fields P_k are smooth with bounded derivatives of all orders in a neighbourhood U of M_c in $\mathbb{R}^{n+1}$. Therefore, there exists a unique strong solution of the SDE (3) in $\mathbb{R}^{n+1}$ that is at first defined up to the first exit time from U.

a) Prove that almost surely, the solution stays on the submanifold M_c for all times, i.e., $X_t \in M_c$ for any $t \in [0, \infty)$.

b) Prove that the SDE (3) can be written in Itô form as

$$dX_t \quad = \quad P(X_t)\ dB_t - \frac{n}{2}\,\kappa(X_t)\,\mathfrak{n}(X_t)\ dt$$

where $\kappa(x) = \frac{1}{n} \operatorname{div} \mathfrak{n}(x)$ is the mean curvature of M_c at x.

Hint: You may use that $\mathfrak{n} \cdot \nabla \mathfrak{n} = 0$ and $\mathfrak{n} \cdot \nabla P_k + P_k \cdot \nabla \mathfrak{n} = 0$. Why do these identities hold ?