Institut für angewandte Mathematik
Summer Semester 2015
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"Stochastic Analysis", Problem sheet 7.

Classes: Monday 12 (0.011), Wednesday 16 (0.006), s6kabash@uni-bonn.de.
Please hand in solutions before Wednesday (!!) June 3, 16.00 s.t.

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1. (Passage time to a sloping line) Let $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{A}, P)$ be a one-dimensional Brownian motion with $X_0 = 0$, and let $a > 0$.

a) Recall that by the reflection principle, the law of the first passage time $T_a = \inf\{t \geq 0 : X_t = a\}$ is absolutely continuous with density

$$f_{T_a}(t) \;=\; at^{-3/2}\,\varphi(a/\sqrt{t})\,1_{(0,\infty)}(t).$$

Here φ denotes the standard normal density.

b) For $a, b \in \mathbb{R}$ with $a > 0$ let $T_L \;=\; \inf\{t \geq 0 : X_t = a + bt\}$ denote the first passage time to the line $y = a + bt$. Show that

$$P[T_L \leq t] \;=\; E_P\left[e^{-bX_t - b^2t/2}; T_a \leq t\right] \;=\; \int_0^t e^{-ab-b^2s/2}\,as^{-3/2}\varphi\left(a/\sqrt{s}\right)\,ds.$$

Conclude that the law of T_L is absolutely continuous with density

$$f_{T_L}(t) \;=\; at^{-3/2}\,\varphi\left((a+bt)/\sqrt{t}\right)\,1_{(0,\infty)}(t).$$

c) Show that for $b > 0$,

$$E_P\left[e^{-bX_t}\max_{s\leq t} X_s\right] \simeq \frac{1}{2b}e^{b^2t/2} \quad \text{and} \quad E_P\left[e^{bX_t}\max_{s\leq t} X_s\right] \simeq bte^{b^2t/2} \qquad \text{as } t \to \infty.$$

2. (Brownian motion writes your name) Prove that Brownian motion in $\mathbb{R}^2$ will write your name (in cursive script, without dotted i's or crossed t's).
To get the pen rolling, first take B_t to be a two-dimensional Brownian motion on $[0, 1]$, and note that for any $[a, b] \subset [0, 1]$ the process

$$X_t^{(a,b)} = (b-a)^{-1/2}(B_{a+t(b-a)} - B_a)$$

is again a Brownian motion on $[0, 1]$. Now, take $g : [0, 1] \to \mathbb{R}^2$ to be a parametrization of your name, and note that Brownian motion spells your name (to precision ϵ) on the interval $[a, b]$ if

$$\sup_{0\leq t\leq 1} |X_t^{a,b} - g(t)| \leq \epsilon. \tag{1}$$

a) Let A_k denote the event that inequality (1) holds for $a = 2^{-k-1}$ and $b = 2^{-k}$. Check that the events A_k are independent, and that one has $P[A_k] = P[A_1]$ for all k. Conclude that if $P[A_1] > 0$ then infinitely many of the A_k will occur with probability one.

b) Show that

$$P\left[\sup_{0\le t\le 1} |B_t| \le \epsilon\right] > 0. \tag{2}$$

c) Finally, complete the solution of the problem by using (2) and Girsanov's Theorem to show that $P[A_1] > 0$; that is to prove

$$P\left[\sup_{0\le t\le 1} |B_t - g(t)| \le \epsilon\right] > 0.$$

3. (Brownian bridge) Let (X_t) be a Brownian motion on $\mathbb{R}^d$ with $X_0 = 0$.

a) Show that for any $y \in \mathbb{R}^d$, the process

$$X_t^y \;=\; X_t - t\cdot(X_1 - y), \qquad t\in[0,1],$$

is independent of X_1.

b) Let $\mu_{0,y}$ denote the law of X^y on $C([0,1],\mathbb{R}^d)$. Show that $y \mapsto \mu_{0,y}$ is a regular version of the conditional distribution of X given $X_1 = y$.