Institut für angewandte Mathematik
Summer Semester 2015
Andreas Eberle

"Stochastic Analysis", Problem sheet 6.

Classes: Monday 12 (0.011), Wednesday 16 (0.006), s6kabash@uni-bonn.de.
Please hand in solutions before Thursday May 21, 15 c.t.

---

1. (Feynman-Kac formula for Itô diffusions)

a) Consider a solution $X_t : \Omega \to \mathbb{R}^n$ of a stochastic differential equation

$$dX_t \quad = \quad b(t, X_t)\ dt + \sigma(t, X_t)\ dB_t, \qquad X_0 = x,$$

with continuous coefficients driven by an $(\mathcal{F}_t)$ Brownian motion taking values in $\mathbb{R}^d$. Fix $t \in (0, \infty)$, and suppose that $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $V : [0, t] \times \mathbb{R}^n \to [0, \infty)$ are continuous functions. Show that if $u \in C^2((0, t] \times \mathbb{R}^n) \cap C([0, t] \times \mathbb{R}^n)$ is a bounded solution of the equation

$$\begin{aligned} \frac{\partial u}{\partial s}(s, x) &= (\mathcal{L}u)(s, x) - V(s, x)u(s, x) \qquad \text{for } s \in (0, t],\ x \in \mathbb{R}^n, \\ u(0, x) &= \varphi(x), \end{aligned}$$

then u has the stochastic representation

$$u(t, x) \;=\; E_x\left[\varphi(X_t)\ \exp\left(-\int_0^t V(t - s, X_s)\, ds\right)\right].$$

Here $\mathcal{L}$ is the generator of X given by

$$(\mathcal{L}F)(t, x) = \frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(t, x)\frac{\partial^2 F}{\partial x^i \partial x^j}(t, x) + \sum_{i=1}^{n} b^i(t, x)\frac{\partial F}{\partial x^i}(t, x) \ \text{ with } a = \sigma\sigma^T.$$

Hint: Consider the time reversal $\hat{u}(s, x) := u(t - s, x)$ of u on $[0, t]$. Show first that $M_r := \exp(-A_r)\hat{u}(r, X_r)$ is a local martingale if $A_r := \int_0^r V(t - s, X_s)\, ds$.

b) The price of a security is modeled by geometric Brownian motion (X_t) with parameters $\alpha, \sigma > 0$. At a price x we have a cost $V(x)$ per unit of time. The total cost up to time t is then given by

$$A_t \;=\; \int_0^t V(X_s) ds\ .$$

Suppose that u is a bounded solution to the PDE

$$\begin{aligned} \frac{\partial}{\partial t}u(t, x) &= \mathcal{L}u(t, x) - \beta V(x)u(t, x), \\ u(0, x) &= 1, \end{aligned}$$

where

$$\mathcal{L} \;=\; \frac{\sigma^2}{2}\, x^2\, \frac{d^2}{dx^2} \;+\; \alpha\, x\, \frac{d}{dx}\;.$$

Show that the Laplace transform of A_t is given by $E_x\left[e^{-\beta A_t}\right] = u(t,x)$.

2. (Brownian motion on the unit sphere) Let $Y_t = B_t/|B_t|$ where $(B_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^n$, $n > 2$. Prove that the time-changed process

$$Z_a = Y_{T_a}, \qquad T = A^{-1} \;\text{ with }\; A_t = \int_0^t |B_s|^{-2}\, ds\;,$$

is a diffusion taking values in the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ with generator

$$\mathcal{L}f(x) \;=\; \frac{1}{2}\left(\Delta f(x) - \sum_{i,j} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right) - \frac{n-1}{2}\sum_i x_i \frac{\partial f}{\partial x_i}(x), \qquad x \in S^{n-1}.$$

3. (Exit distributions for compound Poisson processes)
Let $(X_t)_{t\geq 0}$ be a compound Poisson process with $X_0 = 0$ and jump intensity measure $\nu = N(m,1),\; m > 0$.

a) Determine $\lambda \in \mathbb{R}$ such that $\exp(\lambda X_t)$ is a local martingale.

b) Prove that for $a < 0$,

$$P[T_a < \infty] \;=\; \lim_{b\to\infty} P[T_a < T_b] \;\leq\; \exp(2ma),$$

where

$$T_a := \inf\{t \geq 0 : X_t \leq a\} \qquad \text{and} \qquad T_b := \inf\{t \geq 0 : X_t \geq b\}.$$

Why is it not as easy as for Bessel processes (see the last problem sheet) to compute the ruin probability $P[T_a < T_b]$ exactly?