Institut für angewandte Mathematik
Summer Semester 2015
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“Stochastic Analysis”, Problem sheet 5.

Classes: Monday 12 (0.011), Wednesday 16 (0.006), s6kabash@uni-bonn.de.
Please hand in solutions before Wednesday (!!) May 13, 16.00 st.

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1. (Weak solutions and the martingale problem) Consider a solution of an SDE of the form

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t,$$

driven by a d-dimensional Brownian motion B and with locally bounded coefficients $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$. The corresponding infinitesimal generator is given by

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i \frac{\partial}{\partial x^i}$$

where $a = \sigma\sigma^T$. By Theorem 2.3 in the lecture notes the following are equivalent:

(i) For any $f \in C^2(\mathbb{R}^d)$, the process $M^f = f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)\,ds$ is a local martingale.

(ii) For any $v \in \mathbb{R}^d$, the process $M^v = v \cdot \left(X_t - X_0 - \int_0^t b(X_s)\,ds\right)$ is a local martingale with quadratic variation

$$[M^v]_t = \int_0^t v \cdot a(X_s)v\,ds.$$

a) Show that these conditions are also equivalent to:

(iii) For any $v \in \mathbb{R}^d$, the process

$$Z_t^v := \exp\left(v \cdot \left(X_t - X_0 - \int_0^t b(X_s)\,ds\right) - \frac{1}{2}\int_0^t v \cdot a(X_s)v\,ds\right).$$

is a local martingale.

Hint: In order to prove that (iii) implies (i) it is enough to consider functions of the form $f(y) = \exp(v \cdot y)$. The general case follows, since linear combinations of exponentials are dense in C^2 w.r.t. uniform convergence on compact sets of the functions and their first two derivatives (- this may be assumed without proof).

b) Further, show that these conditions imply that

$$(f(X_t)/f(X_0))\exp\left(-\int_0^t (\mathcal{L}f/f)(X_s)\,ds\right)$$

is a local martingale for any strictly positive C^2 function f.

2. (Variation of constants) We consider nonlinear stochastic differential equations of the form

$$dX_t = f(t, X_t)\,dt + c(t)X_t\,dB_t, \qquad X_0 = x,$$

where $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $c : \mathbb{R}^+ \to \mathbb{R}$ are continuous (deterministic) functions. Proceed as follows :

a) Find an explicit solution Z_t of the equation with $f \equiv 0$.

b) To solve the equation in the general case, use the Ansatz

$$X_t \;=\; C_t \cdot Z_t\ .$$

Show that the SDE gets the form

$$\frac{dC_t(\omega)}{dt} \;=\; f(t, Z_t(\omega) \cdot C_t(\omega))/Z_t(\omega)\ ; \qquad C_0 = x. \tag{1}$$

Note that for each $\omega \in \Omega$, this is a *deterministic* differential equation for the function $t \mapsto C_t(\omega)$. We can therefore solve (1) with ω as a parameter to find $C_t(\omega)$.

c) Apply this method to solve the stochastic differential equation

$$dX_t = \frac{1}{X_t}\,dt + \alpha X_t\,dB_t\ ; \qquad X_0 = x > 0\ ,$$

where α is constant.

d) Apply the method to study the solution of the stochastic differential equation

$$dX_t = X_t^\gamma\,dt + \alpha X_t\,dB_t\ ; \qquad X_0 = x > 0\ ,$$

where α and γ are constants. For which values of γ do we get explosion?

3. (Exit distributions for Bessel processes) Let (X_t) be a solution of the equation

$$dX_t \;=\; \frac{d-1}{2}\,\frac{1}{X_t}\,dt \,+\, dB_t, \qquad X_0 = x_0 > 0,$$

where (B_t) is a standard Brownian motion and $d > 1$ is a constant.

i) Find a non-constant function $u : \mathbb{R} \to \mathbb{R}$ such that $u(X_t)$ is a local martingale.

ii) Compute the ruin probability $P[T_a < T_b]$ for $0 < a < b$ with $x_0 \in [a, b]$, where

$$T_a := \inf\{t \geq 0 : X_t \leq a\} \qquad \text{and} \qquad T_b := \inf\{t \geq 0 : X_t \geq b\}.$$

iii) Proceeding similarly, determine the mean exit time $E[T]$, where $T = \min\{T_a, T_b\}$.