Institut für angewandte Mathematik
Summer Semester 2015
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"Stochastic Analysis", Problem sheet 4.

Classes: Monday 12 (0.011), Wednesday 16 (0.006), s6kabash@uni-bonn.de.
Please hand in solutions before Thursday May 7, 15 ct, at the mailbox opposite to the library entrance.

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1. (Jumps and structure of Lévy processes)

a) Show that the probability that a Lévy process (X_t) jumps at a given fixed time t is zero.

b) Suppose (X_t) is a Lévy martingale without Brownian component (i.e. $\sigma = 0$ in the Lévy-Ito decomposition). Show that if the total jump intensity $\lambda = \nu(\mathbb{R})$ is finite, then X is a compensated compound Poisson process with jump intensity measure ν.

c) Conclude that any Lévy martingale without Brownian component is a limit of compensated compound Poisson processes.

2. (Geometric Poisson processes and change of measure) Let $(N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$ on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t\geq 0}$.

a) Let $\sigma, \alpha \in \mathbb{R}$ with $\sigma > -1$. Give a meaning to the SDE

$$dS_t \;=\; \sigma\, S_{t-}\, dN_t \,+\, \alpha\, S_t\, dt\,, \qquad S_0 = 1,$$

and find a solution by the ansatz $S_t = \exp\left(aN_t + bt\right)$.

b) Given σ, for which value of α is (S_t) a martingale ?

c) Now let $\mu > 0$. Verify that

$$Z_t \;=\; (\mu/\lambda)^{N_t}\, e^{(\lambda-\mu)t}$$

is an $(\mathcal{F}_t)$ martingale with $E[Z_t] = 1$ for all t.

d) We define a new probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}_1)$ by

$$\tilde{\mathbb{P}}[A] \;=\; \int_A Z_1\, d\mathbb{P} \qquad \text{for any } A \in \mathcal{F}_1.$$

Verify that $\tilde{E}[X_t] = E[X_t Z_t]$ for any $\mathcal{F}_t$ measurable random variable X_t and $t \in [0,1]$. Compute the characteristic function of the process $(N_t)_{t\in[0,1]}$ w.r.t. the new measure $\tilde{\mathbb{P}}$. Conclude that under $\tilde{\mathbb{P}}$, (N_t) is a Poisson process with intensity μ.

3. (Simulation of Lévy processes) Write a program that simulates and plots a trajectory of a Lévy jump process with finite jump intensity measure (e.g. with Mathematica).