Institut für angewandte Mathematik
Summer Semester 2015
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"Stochastic Analysis", Problem sheet 3.

Classes: Monday 12 (0.011), Wednesday 16 (0.006), s6kabash@uni-bonn.de.
Please hand in solutions before Thursday 30th, 15 ct, at the mailbox opposite to the library entrance.

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1. (Ito isometries) Let $(\mathcal{F}_t)$ be a filtration on a probability space $(\Omega, \mathcal{A}, P)$.

a) As a warm-up, prove that the Ito isometry

$$\mathbb{E}\left[\left(\int_0^t G_s\, dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t G_s^2\, ds\right]$$

holds for stochastic integrals of elementary $(\mathcal{F}_t)$ predictable processes w.r.t. an $(\mathcal{F}_t)$ Brownian motion.

b) More generally, suppose that M_t is a square integrable $(\mathcal{F}_t)$ martingale, and $\langle M\rangle_t$ is an increasing predictable càdlàg process such that $M_t^2 - \langle M\rangle_t$ is a martingale. Prove that the Ito isometry

$$\mathbb{E}\left[\left(\int_0^t G_s\, dM_s\right)^2\right] = \mathbb{E}\left[\int_0^t G_s^2\, d\langle M\rangle_s\right]$$

holds for elementary predictable processes G. Which form does the Ito isometry take if M_t is a Lévy process ?

c) Use b) to define the stochastic integral w.r.t. a square integrable Lévy martingale for an arbitrary predictable integrand $G \in L^2(P \otimes \lambda_{(0,t)})$. *(You may assume without proof that the elementary predictable processes are dense in the space of square integrable predictable processes $(G_s)_{s\in(0,t)}$).*

2. (Quadratic variation and covariation) Let $(\pi_n)_{n\in\mathbb{N}}$ be sequence of partitions of $\mathbb{R}_+$ with $\mathrm{mesh}(\pi_n) \to 0$. The covariation of two functions $X, Y : [0,\infty) \to \mathbb{R}$ is defined as

$$[X,Y]_t = \lim_{n\to\infty} \sum_{s\in\pi_n} (X_{s'\wedge t} - X_{s\wedge t})(Y_{s'\wedge t} - Y_{s\wedge t}) \qquad \text{for any } t \geq 0$$

with $s' = \min\{u \in \pi_n : u > s\}$, provided the limit exists. The quadratic variation of X is $[X]_t = [X, X]_t$. Now suppose that (B_t) is a Brownian motion, and (N_t) is an independent Poisson process on a probability space $(\Omega, \mathcal{A}, P)$.

a) Prove that $[B]_t = t$ holds P-almost surely if $\sum_n \mathrm{mesh}(\pi_n) < \infty$.

b) Show that $[N]_t = N_t$.

c) Compute $[B, N]_t$.

3. (Expectation values and martingales for Poisson point processes with infinite intensity) Let (N_t) be a Poisson point process with σ-finite intensity measure ν.

a) By considering at first elementary functions, prove that for $t \geq 0$, the identity

$$E\left[\int f(y) N_t(dy)\right] \quad = \quad t \int f(y)\nu(dy)$$

holds for any measurable function $f : S \to [0, \infty]$. Conclude that for $f \in \mathcal{L}^1(\nu)$, the integral $N_t(f) = \int f(y) N_t(dy)$ exists almost surely and defines a random variable in $L^1(\Omega, \mathcal{A}, P)$.

b) Proceeding similarly as in a), prove the identities

$$\begin{aligned} E[N_t(f)] &= t \int f \, d\nu \qquad \text{for any } f \in \mathcal{L}^1(\nu), \\ \mathrm{Cov}[N_t(f), N_t(g)] &= t \int fg \, d\nu \qquad \text{for any } f, g \in \mathcal{L}^1(\nu) \cap \mathcal{L}^2(\nu), \\ E[\exp(ipN_t(f))] &= \exp(t \int (e^{ipf} - 1) \, d\nu) \qquad \text{for any } f \in \mathcal{L}^1(\nu). \end{aligned}$$

c) Show that the processes considered in Corollary 1.11 are again martingales provided $f \in \mathcal{L}^1(\nu)$, $f, g \in \mathcal{L}^1(\nu) \cap \mathcal{L}^2(\nu)$ respectively.