Institut für angewandte Mathematik
Summer Semester 2015
Andreas Eberle

“Stochastic Analysis”, Problem sheet 2.

Classes: Monday 12 (0.011), Wednesday 16 (0.006), s6kabash@uni-bonn.de. Please hand in solutions for Exercise 1-3 before Monday 20th, 11 ct, and solutions for Exercise 4 and 5 before Thursday 23rd, 15 ct, at the mailbox opposite to the library entrance.

---

1. (Martingales of compound Poisson processes) Consider a compound Poisson process given by

$$X_t \;=\; \sum_{i=1}^{N_t} Y_i, \qquad t \geq 0,$$

with a Poisson process (N_t) of intensity $\lambda > 0$ and independent i.i.d. random variables Y_i, $i \in \mathbb{N}$, with distribution ν, expectation value m and finite variance σ^2.

a) Prove that (X_t) is a Lévy process with characteristic exponent

$$\psi(p) \;= \lambda \int (1 - \exp(ip \cdot y))\, \nu(dy)\,.$$

b) Show that $M_t := X_t - m\lambda t$ is a martingale.

c) Suppose that ν is a normal distribution. For which values of λ is the process

$$Z_t \;=\; \exp(-X_t + (m - \frac{1}{2}\sigma^2)t)$$

a supermartingale? Consider the cases $m = \sigma^2/2$, $m < \sigma^2/2$ and $m > \sigma^2/2$.

2. (Sample paths of Inverse Gaussian processes) Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion with $B_0 = 0$.

a) Show that the paths of the process $S_t := \sup_{s\leq t} B_s$ are almost surely not strictly increasing on any non-empty open interval $(a, b) \subset [0, \infty)$.

b) Let

$$T_s := \inf\{t \geq 0 : \; B_t = s\}$$

denote the first passage time to a level $s \in \mathbb{R}$. Prove that the process $(T_s)_{s\geq 0}$ is increasing and purely discontinuous, i.e., with probability one, (T_s) is not continuous on any non-empty open interval $(a, b) \subset [0, \infty)$.

3. (Strong Markov property for Lévy processes) Let $(X_t)_{t\geq 0}$ be a Lévy process w.r.t. the filtration $(\mathcal{F}_t)_{t\geq 0}$ and let T be a finite stopping time. Show that $Y_t = X_{T+t} - X_T$ is a process that is independent of $\mathcal{F}_T$, and X and Y have the same law.

Hint: Consider the sequence of stopping times $(T_n)_n$ defined by

$$T_n(\omega) = \frac{k+1}{2^n} \qquad \text{if} \quad \frac{k}{2^n} \leq T < \frac{k+1}{2^n}.$$

Notice that $T_n \downarrow T$ as $n \to \infty$. In a first step show that for any $t_1 < t_2 < \ldots < t_m$, $m \geq 1$, any bounded continuous function f on $\mathbb{R}^m$, and any $A \in \mathcal{F}_T$ we have

$$E\left[f(X_{T_n+t_1} - X_{T_n}, \ldots, X_{T_n+t_m} - X_{T_n}) 1_A\right] = E\left[f(X_{t_1}, \ldots, X_{t_m})\right] P[A].$$

4. (A characterization of Poisson processes) Let $(X_t)_{t\geq 0}$ be a Lévy process with $X_0 = 0$ a.s. Suppose that the paths of X are piecewise constant, increasing, all jumps of X are of size 1, and X is not identically 0. Prove that X is a Poisson process.

Hint: Apply the Strong Markov property in Problem 3 to the jump times $(T_i)_{i=1,2,\ldots}$ of X to conclude that the random variables $U_i := T_i - T_{i-1}$ are i.i.d. (with $T_0 := 0$). Then, it remains to show that U_1 is an exponential random variable with some parameter $\lambda > 0$.

5. (Construction of Poisson point processes) Let $(S, \mathcal{S}, \nu)$ be a σ-finite measure space, and let $\lambda := \nu(S)$.

a) Suppose first that $\lambda \in (0, \infty)$. Prove that

$$N_t = \sum_{j=1}^{K_t} \delta_{\eta_j}, \qquad t \geq 0,$$

is a Poisson point process with intensity measure ν provided the random variables η_j, $j \in \mathbb{N}$, are independent with distribution $\lambda^{-1}\nu$, and (K_t) is an independent Poisson process of intensity λ.

b) Now consider the case $\lambda = \infty$. Let $(\nu_k)_{k\in\mathbb{N}}$ be a sequence of finite measures on $(S, \mathcal{S})$ with $\nu = \sum \nu_k$. Prove that if $(N_t^{(k)})_{t\geq 0}$, $k \in \mathbb{N}$, are independent Poisson point processes on $(S, \mathcal{S})$ with intensity measures ν_k then

$$\overline{N}_t = \sum_{k=1}^{\infty} N_t^{(k)}$$

is a Poisson point process with intensity measure ν.

Hint: You may use the subdivision and superposition property of Poisson processes.