Institut für angewandte Mathematik
Summer Semester 2015
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"Stochastic Analysis", Problem sheet 1.

Classes: Monday 12 (0.011), Wednesday 16 (0.006), s6kabash@uni-bonn.de.
Please hand in your solutions before Monday 13th, 11 ct, at the mailbox opposite to the library entrance.

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1. (Brownian motion as a random Fourier series)

a) Compute a complete orthonormal basis of $L^2(0,1)$ consisting of eigenfunctions of the operator d^2/dt^2 with Dirichlet boundary conditions at 0 and 1.

b) Now consider a one-dimensional Brownian motion B_t with $B_0 = 0$. Prove the representation

$$B_t \;=\; B_1 \cdot t + \sum_{k=1}^{\infty} Z_k \, \frac{\sqrt{2}\cdot \sin(k\pi t)}{k\pi} \qquad \text{for } t \in [0,1]$$

with independent standard normal random variables $Z_1, Z_2, \ldots$. In which sense does the series converge ?

2. (Martingales and hitting times for Poisson processes) Let (N_t) be a Poisson process with intensity $\lambda > 0$ and $N_0 = 0$.

a) Prove that the compensated Poisson process $M_t = N_t - \lambda_t$ and $M_t^2 - \lambda t$ are $(\mathcal{F}_t^N)$ martingales.

b) Let $T_a = \inf\{t \geq 0 : N_t = a\}$, where a is a positive integer. Show that

$$E[T_a] \;=\; \frac{a}{\lambda} \qquad \text{and} \qquad \mathrm{Var}\,[T_a] \;=\; \frac{a}{\lambda^2}.$$

3. (Properties of càdlàg functions)

a) Prove that if I is a compact interval, then for any càdlàg function $x : I \to \mathbb{R}$, the set $\{s \in I : |\Delta x_s| > \varepsilon\}$ is finite for any $\varepsilon > 0$. Conclude that any càdlàg function $x : [0,\infty) \to \mathbb{R}$ has at most countably many jumps.

b) Show that a uniform limit of a sequence of càdlàg functions is again càdlàg.