**Institut für angewandte Mathematik**
**Summer Semester 2015**
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# “Stochastic Analysis”, Problem sheet 10.

**Please hand in solutions before Thursday July 2, 15 ct.**

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**1. (Order of Convergence)** Let $(X_t)_{t\geq 0}$ be an $n$-dimensional stochastic process satisfying the SDE

$$dX_t \;=\; b(X_t)\,dt + \sum_{k=1}^{d} \sigma_k(X_t)\,dB_t^k,$$

where $b, \sigma_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, \ldots, d$, are bounded continuous functions, and $B$ is a $d$-dimensional Brownian motion. Prove that as $h \downarrow 0$,

a) $X_{t+h}$ converges to $X_t$ with strong $L^2$ order 1/2.

b) $X_{t+h}$ converges to $X_t$ with weak order 1.

**2. (Convergence of Riemann sum approximations to stochastic integrals)** Let $(M_t)$ and $(H_t)$ be a bounded continuous $(\mathcal{F}_t)$ martingale, and a bounded continuous $(\mathcal{F}_t)$ adapted process respectively, and fix $a \in (0, \infty)$. For a partition $\pi$ of $\mathbb{R}^+$ let

$$I_t^\pi \;:=\; \sum_{\substack{s\in\pi \\ s<t}} H_s(M_{s'\wedge t} - M_s)$$

denote the left Riemann sum approximation to the stochastic integral $I_t = \int_0^t H_s\, dM_s$.

a) Show without assuming the existence of the quadratic variation of $(M_t)$ that for every $\gamma > 0$ there exists a constant $\Delta > 0$ such that

$$\|I^\pi - I^{\tilde{\pi}}\|^2_{M^2([0,a])} < \gamma$$

holds for any partitions $\pi$ and $\tilde{\pi}$ of $\mathbb{R}_+$ with $\mathrm{mesh}(\pi) < \Delta$ and $\mathrm{mesh}(\tilde{\pi}) < \Delta$.

*Remark: The assertion carries over to càdlàg processes, see Theorem 5.7 in the lecture notes.*

b) Conclude that the stochastic integral $I_t$ and the quadratic variation $[M]_t$ exist along any sequence $(\pi_n)$ of partitions with mesh $(\pi_n) \to 0$, and $I$ and $[M]$ are continuous processes that do not depend on the chosen partition sequence.

c) Define the stochastic integral of an arbitrary continuous adapted process w.r.t. a continuous semimartingale.

**3. (Lévy Area)** If $c(t) = (x(t), y(t))$ is a smooth curve in $\mathbb{R}^2$ with $c(0) = 0$, then

$$A(t) \;=\; \int_0^t (x(s)y'(s) - y(s)x'(s))\,ds \;=\; \int_0^t x\,dy - \int_0^t y\,dx$$

describes the area that is covered by the secant from the origin to $c(s)$ in the interval $[0, t]$. Analogously, for a two-dimensional Brownian motion $B_t = (X_t, Y_t)$ with $B_0 = 0$, one defines the *Lévy Area*

$$A_t \;:=\; \int_0^t X_s\,dY_s \;-\; \int_0^t Y_s\,dX_s\,.$$

a) Let $\alpha(t)$, $\beta(t)$ be $C^1$-functions, $p \in \mathbb{R}$, and

$$V_t \;=\; ipA_t \;-\; \frac{\alpha(t)}{2}\left(X_t^2 + Y_t^2\right) \;+\; \beta(t)\,.$$

Show using Itô's formula, that $e^{V_t}$ is a local martingale provided $\alpha'(t) = \alpha(t)^2 - p^2$ and $\beta'(t) = \alpha(t)$.

b) Let $t_0 \in [0, \infty)$. The solutions of the ordinary differential equations for $\alpha$ and $\beta$ with $\alpha(t_0) = \beta(t_0) = 0$ are

$$\begin{aligned}\alpha(t) &= p \cdot \tanh(p \cdot (t_0 - t))\,,\\ \beta(t) &= -\log\cosh(p \cdot (t_0 - t))\,.\end{aligned}$$

Conclude that

$$E\left[e^{ipA_{t_0}}\right] \;=\; \frac{1}{\cosh(pt_0)} \qquad \forall\, p \in \mathbb{R}\,.$$

c) Show that the distribution of $A_t$ is absolutely continuous with density

$$f_{A_t}(x) \;=\; \frac{1}{2t\cosh(\frac{\pi x}{2t})}\,.$$