7.1. MALLIAVIN GRADIENT AND INTEGRATION BY PARTS ON WIENER SPACE

$\Omega = C([0,1], \mathbb{R}^d)$ with sup-norm

P = Wiener measure (start in 0)

$B_t(\omega) = \omega(t)$ canonical BM

<u>DEF.</u> $F: \Omega \to \mathbb{R}$ <u>Fréchet differentiable at ω</u>

$:\Leftrightarrow$ $\exists$ contin. linear form $d_\omega F : \Omega \to \mathbb{R}$:

$$\| F(\omega+h) - F(\omega) - (d_\omega F)(h) \|_\Omega = o(\|h\|_\Omega) \quad \forall h \in \Omega$$

In this case there exist directional derivatives

$$\frac{\partial F}{\partial h}(\omega) = \lim_{\varepsilon \downarrow 0} \frac{F(\omega + \varepsilon h) - F(\omega)}{\varepsilon} = (d_\omega F)(h) \quad \forall h \in \Omega$$

<u>PROBLEM</u>: X_t sol. of s.d.e. is in general <u>not</u> Fréchet-diff.

E.g. $F = \int_0^1 B_t^1 \, dB_t^2$: $\frac{\partial F}{\partial h} = \int_0^1 h_t^1 \, dB_t^2 + \int_0^1 B_t^1 \, dh_t^2$ formal derivative

NOT CONTIN. in h w.r.t. sup-norm!

$$H = \{ h : [0,1] \to \mathbb{R}^d \mid h(0) = 0,\ h \text{ abs. contin. with } h' \in L^2 \}$$

$$(h, g)_H = \int_0^1 h_t' g_t' \, dt$$

CAMERON-MARTIN SPACE ($\to$ tangent space)

DEF. (MALLIAVIN GRADIENT I)

1) For $F \in C_b^1(\Omega)$ and $\omega \in \Omega$ define $(D^H F)(\omega) \in H$ via

$$(*) \quad \left((D^H F)(\omega), h \right)_H = \frac{\partial F}{\partial h}(\omega) = (d_\omega F)(h) \quad \forall h \in H,$$

i.e., $D^H F$ is gradient of F w.r.t. metric $(\cdot,\cdot)_H$

2) $$(D_t F)(\omega) := \frac{d}{dt} (D^H F)(\omega) \in L^2([0,1], \mathbb{R}^d)$$

EXAMPLES 1) Brownian motion: $\boxed{(D_t B_s^i)(\omega) = e_i \, I_{(0,s)}(t)}$

2) Wiener integrals: $F = \int_0^1 g \cdot dB$, $g \in C^1([0,1], \mathbb{R}^d)$

$$\Rightarrow \boxed{(D_t F)(\omega) = g(t)}$$

THEOREM (INTEGRATION BY PARTS, BISMUT)

$F \in C_b^1(\Omega)$, $G \in \mathcal{L}_a^2(\Omega \times [0,1], P \otimes dt)$, $H_t = \int_0^t G_s \, ds$

$$(**) \quad \underbrace{E\left[\int_0^1 D_t F \cdot G_t \, dt\right]}_{(DF, G)_{L^2(P\otimes\lambda)}} = E\Big[\overbrace{(D^H F, H)_H}^{=\partial_H F}\Big] = \underbrace{E\left[F \int_0^1 G_t \, dB_t\right]}_{(F, \int_0^1 G \, dB)_{L^2(P)}}$$

REMARK: Infinitesimal version of Girsanov theorem:

$B = (B_t)_{0 \le t \le 1}$ identity map on Ω

$E[F(B + \varepsilon H)] = E[F(B) \cdot Z_1^\varepsilon]$ where

$$Z_t^\varepsilon := \exp\left(\varepsilon \int_0^t G_s \cdot dB_s - \frac{\varepsilon^2}{2} \int_0^t |G_s|^2 \, ds\right).$$

Derivative w.r.t. ε $\Rightarrow$ Claim

<u>PROOF</u>: 1) Assume G is bounded. Then by Novikov, Z_t^ε is a martingale, and by Girsanov,

$$E\left[\frac{F(B+\varepsilon H)-F(B)}{\varepsilon}\right] = E\left[F(B)\,\frac{Z_1^\varepsilon - 1}{\varepsilon}\right]$$

$\varepsilon \downarrow 0$: r.h.s. $\to E\left[F(B)\int_0^1 G\,dB\right]$

since $\dfrac{Z_1^\varepsilon - 1}{\varepsilon} = \int_0^1 \underbrace{Z^\varepsilon}_{\to 1 \text{ in } L^2(P\otimes\lambda) \text{ (Exercise)}} G\,dB \xrightarrow{\varepsilon\downarrow 0} \int_0^1 G\,dB$ in $L^2(P)$.

$dZ^\varepsilon = \varepsilon Z^\varepsilon G\,dB$

$$\text{l.h.s.} = E\Big[\frac{1}{\varepsilon}\int_0^\varepsilon \underbrace{(\partial_H F)(B+sH)}_{\substack{\to (\partial_H F)(B) \text{ as } s\downarrow 0\\ \text{since } F\in C_b^1(\Omega)}}\,ds\Big] \xrightarrow{\text{Leb.}} E\left[(\partial_H F)(B)\right]$$

2) Since both sides of (**) are continuous in G w.r.t. the $L^2(P\otimes\lambda)$ norm, the equation extends to all $G\in\mathcal{L}_a^2(P\otimes\lambda)$. $\square$

A FIRST APPLICATION: SKOROKHOD INTEGRAL

$$D: \quad \begin{array}{c} F \longmapsto (D_t F)_{0 \le t \le 1} \\ C_b^1(\Omega) \subseteq L^2(\Omega, P) \longrightarrow L^2(\Omega \times [0,1], P \otimes \lambda) \end{array}$$

is a densely defined linear operator. Define

$$\delta : \mathrm{Dom}(\delta) \subseteq L^2(P \otimes \lambda) \to L^2(P)$$

as the adjoint operator (divergence). Then:

$$G \in \mathcal{L}_a^2(P \otimes \lambda) \overset{\text{Bismut}}{\Longrightarrow} G \in \mathrm{Dom}(\delta), \quad \delta(G) = \int_0^1 G \, dB$$

Hence δ is an <u>extension of the Itô integral</u> to not necessarily adapted processes $G \in L^2(P \otimes \lambda)$!

<u>EXAMPLE</u> G_t adapted + bounded, $F \in C_b^1(\Omega)$

$$\Rightarrow (F \cdot G_t)_{0 \le t \le 1} \in \mathrm{Dom}(\delta)$$

Skorokhod integral: $\delta(FG) = F\,\delta(G) - \int_0^1 D_t F \, G_t \, dt$

"product rule for divergence operator"

DEFINITION OF MALLIAVIN GRADIENT II (CLOSURE)

$\mathbb{D}^{1,p} :=$ closure of $C_b^1(\Omega) \subseteq L^p(\Omega;P)$ w.r.t. norm

$$\|F\|_{1,p} := E\left[|F|^p + \|D^H F\|_H^p \right]^{\frac{1}{p}} , \quad p \in [1,\infty)$$

THEOREM 1) $\exists!$ extension of D^H to a contin. lin. op.

$$D^H : \mathbb{D}^{1,p} \subseteq L^p(\Omega;P) \to L^p(\Omega \to H; P)$$

2) The Bismut integration by parts formula holds for all $F \in \mathbb{D}^{1,2}$.

THEOREM (PRODUCT- AND CHAIN RULE)

1) $F, G \in \mathbb{D}^{1,2} \cap L^\infty \Rightarrow F \cdot G \in \mathbb{D}^{1,2}$ with

$$D(F \cdot G) = F\, DG + G\, DF .$$

2) $F_1, \dots, F_m \in \mathbb{D}^{1,2}$, $\phi \in C_b^1(\mathbb{R}^m) \Rightarrow \phi(F_1, \dots, F_m) \in \mathbb{D}^{1,2}$ with

$$D\, \phi(F_1, \dots, F_m) = \sum_{i=1}^m \frac{\partial \phi}{\partial x_i}(F_1, \dots, F_m)\, DF_i$$

7.2. ITÔ REPRESENTATION THEOREM AND CLARK-OCONE FORMULA

$\mathcal{F}_t = \sigma(B_s \mid 0 \le s \le t)^P$ Brownian filtration

THEOREM (ITÔ) For any $F \in \mathcal{L}^2(\Omega, \mathcal{F}_1, P)$ there exists a unique $G \in L^2_a(0,1)$ s.t.

$$F = E[F] + \int_0^1 G_s \, dB_s \quad P\text{-a.s.}$$

COROLLARY For any L^2-bounded $(\mathcal{F}_t)$-martingale $(M_t)_{0 \le t \le 1}$ there exists a unique $G \in L^2_a(0,1)$ s.t.

$$M_t = M_0 + \int_0^t G_s \, dB_s \quad P\text{-a.s.} \quad \forall t \in [0,1].$$

In particular: Every $(\mathcal{F}_t)$-martingale has a continuous modification!

REMARK: Assumption $\mathcal{F}_t = \mathcal{F}_t^{B,P}$ is crucial.

Identification of G_t ?

THEOREM (CLARK-OCONE) For $F \in \mathbb{D}^{1,2}$,

$$F = E[F] + \int_0^1 E[D_t F \mid \mathcal{F}_t]\, dB_t \quad P\text{-a.s.}$$

EXAMPLE: Look-back-option: $F = \max_{0 \le t \le 1} B_t$

Fact: For P-a.e. ω there ex. a unique $T(\omega) \in [0,1]$ s.t. $F = B_T$.

$F \in \mathbb{D}^{1,2}$ with $D_t F(\omega) = I_{[0,T(\omega)]}(t)$.

$$E[D_t F \mid \mathcal{F}_t](\omega) = P[T > t \mid \mathcal{F}_t](\omega)$$

$$= P[\underbrace{\max_{t \le s \le 1} B_s}_{= B_t + (B_s - B_t)} \ge M_t \mid \mathcal{F}_t](\omega) = P[\underbrace{\max_{t \le s \le 1}(B_s - B_t)}_{\sim \max_{0 \le s \le 1-t} B_s \sim |B_{1-t}| \sim \sqrt{1-t}\cdot|B_1|} \ge M_t(\omega) - B_t(\omega)]$$

$$= 2\,\Phi\left(\frac{M_t(\omega) - B_t(\omega)}{\sqrt{1-t}}\right), \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2}\, dy$$

$$\xRightarrow{\text{Clark-Ocone}} \quad M_1 = \sqrt{\frac{2}{\pi}} + \int_0^1 2\,\Phi\left(\frac{M_t - B_t}{\sqrt{1-t}}\right) dB_t$$

7.3. FIRST APPLICATIONS TO SDE

THEOREM (Malliavin derivative of Itô integrals)

Suppose $G \in \mathcal{L}^2_a(P \otimes \lambda_{[0,1]})$ with $G_s \in \mathbb{D}^{1,2}$ for all $s \in [0,1]$ and $(\omega, t, s) \longmapsto (D_t G_s)(\omega) \in L^2(\Omega \times [0,1]^2, P \otimes \lambda^2)$.

Then $\int_0^u G\, dB \in \mathbb{D}^{1,2}$ for all $u \in [0,1]$ and

$$D_t \int_0^u G_s \, dB_s = \int_0^u D_t G_s \cdot dB_s + G_t \cdot I_{(0,u)}(t) \quad P \otimes \lambda\text{-a.s.}$$

$$\boxed{(*)\; dX_t = \sum_{j=1}^d \sigma_j(X_t)\, dB^j_t + b(X_t)\, dt}$$

THEOREM If $\sigma_j, b : \mathbb{R}^n \to \mathbb{R}^n$ are Lipschitz then $X^i_t \in \mathbb{D}^{1,p}$ for all $1 \le i \le n$ and $1 \le p < \infty$ with Malliavin derivative

$$D^k_r X_t = \begin{cases} \sigma_k(X_r) + \sum_j \int_r^t \sigma_j'(X_s)\, D^k_r X_s \, dB^j_s + \int_r^t b'(X_s)\, D^k_r X_s \, ds & \text{for } t \ge r, \\ 0 & \text{for } t < r. \end{cases}$$

Application 1: Variation of the initial condition

ξ_t^x = flow of sde (*)

$Y_t^x = \xi_t'(x)$ derivative flow

is fundamental solution of linear sde

$$(**) \qquad dY_t = \sum_j \sigma_j'(\xi_t) Y_t \, dB_t^j + b'(\xi_t) Y_t \, dt$$

$$Y_0 = I$$

COROLLARY $D_r^k \xi_t = Y_t Y_r^{-1} \sigma_k(\xi_r) I_{\{t \geq r\}}$

Conversely: If σ is strictly elliptic then

$$Y_t = D_r \xi_t \, \sigma(\xi_r)^{-1} Y_r \qquad \forall r \leq t$$

$$\Rightarrow Y_t = \frac{1}{t} \int_0^t D_r \xi_t \, \sigma(\xi_r)^{-1} Y_r \, dr$$

COROLLARY (BISMUT-ELWORTHY) For all $t>0$, $f \in C_b^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$:

$$d_x E[f(\xi_t^x)] = \frac{1}{t} E\left[f(\xi_t^x) \int_0^t \sigma(\xi_r^x)^{-1} Y_r^x \, dB_r\right]$$

Remark: R.h.s. does not involve derivatives!

$\rightsquigarrow$ Numerical schemes for computing l.h.s.

Application 2: Variation of the drift coefficient

$$dX_t^\varepsilon = \sum_j \sigma_j(X_t^\varepsilon)\,dB_t^j + b(\varepsilon, X_t^\varepsilon)\,dt, \quad X_0^\varepsilon = x$$

$\varepsilon \in (-\delta, \delta)$, σ strictly elliptic, $X_t := X_t^0$

FACT: $b : (-\delta,\delta)\times\mathbb{R}^n \to \mathbb{R}^n$, $\sigma_j : \mathbb{R}^n \to \mathbb{R}^n$ $\in C_b^2$

$$\Rightarrow \quad \exists\; V_t := \frac{\partial}{\partial\varepsilon} X_t^\varepsilon \Big|_{\varepsilon=0}, \quad V_0 = 0, \text{ and}$$

$$dV_t = \sum_j \sigma_j'(X_t)V_t\,dB_t^j + b'(0,X_t)V_t\,dt + \frac{\partial b}{\partial\varepsilon}(0,X_t)\,dt$$

COROLLARY 1) $V_t = \int_0^t (D_r X_t)\,\sigma(X_r)^{-1}\frac{\partial b}{\partial \varepsilon}(0,X_r)\,dr$

2) $\frac{\partial}{\partial\varepsilon} E[f(X_t^\varepsilon)]\Big|_{\varepsilon=0} = E\Big[f(X_t)\int_0^t \sigma(X_r)^{-1}\frac{\partial b}{\partial\varepsilon}(0,X_r)\,dB_r\Big]$

<u>Proof:</u> 1) Variation of constants $\Rightarrow$

$$V_t = \int_0^t Y_t Y_r^{-1} \frac{\partial b}{\partial \varepsilon}(0, X_r)\, dr$$

$$= \int_0^t (D_r X_t)\, \sigma(X_r)^{-1} \frac{\partial b}{\partial \varepsilon}(0, X_r)\, dr$$

2) $\frac{\partial}{\partial \varepsilon} E[f(X_t^\varepsilon)]\Big|_{\varepsilon=0} = E[f'(X_t) V_t] = \text{r.h.s.}$

by 1) and the Bismut integration by parts identity. $\square$

<u>Further important applications of Malliavin calculus:</u>

- Existence of (smooth) densities for the distributions of solutions of s.d.e.
- Stochastic proof of Hörmander's theorem (Hypoellipticity)

5.5 EXISTENCE & SMOOTHNESS OF DENSITIES

$$(*) \quad dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t\,, \quad X_0 = x\,.$$

$P_t(x,\cdot) = P\circ X_t^{-1}$ absolutely continuous ?

THEOREM 5.21 (Malliavin's lemma)

Let μ be a finite measure on $\mathcal{B}(\mathbb{R}^n)$.

1) If there exists $C \in \mathbb{R}_+$ s.t.

$$\left|\int \nabla f \, d\mu\right| \leq C \cdot \sup|f| \quad \forall f \in C_b^\infty(\mathbb{R}^n)$$

then $\mu \ll \lambda^n$ with density $\frac{d\mu}{dx} \in L^{\frac{n}{n-1}}(\mathbb{R}^n, dx)$.

2) If for any multi-index $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}_+^n$, there exists $C_\alpha \in \mathbb{R}_+$ s.t.

$$\left|\int \partial^\alpha f \, d\mu\right| \leq C_\alpha \cdot \sup|f| \quad \forall f \in C_b^\infty(\mathbb{R}^n)$$

then $\frac{d\mu}{dx} \in C^\infty(\mathbb{R}^n)$.

Rem. Compare to Sobolev embedding theorem!

Application to SDE (*):

GOAL: $\left|\int \nabla f(y)\, p_t(x,dy)\right| = \left|E[\nabla f(X_t)]\right| \leq C \cdot \sup |f|$

IDEA $(d=n=1)$:

$$D_r f(X_t) = f'(X_t)\, D_r X_t = f'(X_t)\, Y_t Y_r^{-1} \sigma(X_r)\, I_{\{r \leq t\}}$$

If σ is non-degenerate, this implies

$$f'(X_t) = \frac{1}{t}\int_0^t D_r f(X_t)\, \sigma(X_r)^{-1} Y_r\, dr\; Y_t^{-1} = \frac{1}{t}\left(D^H f(X_t), H\right)_H Y_t^{-1}$$

where $H_s = \int_0^{s \wedge t} \sigma(X_r)^{-1} Y_r\, dr$. By taking expectations on both sides and integrating by parts, one can prove:

THEOREM 5.22 (Absolute continuity of transition densities)

Suppose $\sigma, b \in C_b^1$ gradient, and $\sigma(x) \geq \varepsilon \cdot I_n \;\forall x$ with $\varepsilon > 0$. Then $p_t(x,\cdot) \ll \lambda^n$ for any $t > 0$.

EXTENSIONS:
- Higher regularity / smoothness by repeated i.b.p.
- Degenerate diffusion coefficient: Absolute continuity / Smoothness under Hörmander's condition