Institut für angewandte Mathematik
Winter Semester 11/12
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"Stochastic Analysis", Problem sheet 9.

Please hand in the solutions before Tuesday 13.12., 2 pm

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1. (Quadratic variation) Let $(X_t)_{t\geq 0}$ be a strict local martingale, and let S and T be stopping times satisfying $S \leq T$. Show:

$$[X]_S = [X]_T \;\Rightarrow\; X \text{ is a.s. constant on } [S,T].$$

2. (Order of Convergence) Let $(X_t)_{t\geq 0}$ be an n-dimensional stochastic process satisfying the SDE

$$dX_t \;=\; b(X_t)\,dt + \sum_{k=1}^{d} \sigma_k(X_t)\,dB_t^k,$$

where $b, \sigma_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, \ldots, d$, are bounded continuous functions, and B is a d-dimensional Brownian motion. Prove that as $h \downarrow 0$,

a) X_{t+h} converges to X_t with strong L^2 order 1/2, i.e.

$$E[|X_{t+h} - X_t|^2]^{1/2} = \mathcal{O}(h^{1/2}).$$

b) X_{t+h} converges to X_t with weak order 1, i.e.,

$$E[f(X_{t+h})] - E[f(X_t)] = \mathcal{O}(h) \qquad \text{for any } f \in C_b^2.$$

3. (Lévy Area) If $c(t) = (x(t), y(t))$ is a smooth curve in $\mathbb{R}^2$ with $c(0) = 0$, then

$$A(t) \;=\; \int_0^t (x(s)y'(s) - y(s)x'(s))\,ds \;=\; \int_0^t x\,dy - \int_0^t y\,dx$$

describes the area that is covered by the secant from the origin to $c(s)$ in the interval $[0,t]$. Analogously, for a two-dimensional Brownian motion $B_t = (X_t, Y_t)$ with $B_0 = 0$, one defines the *Lévy Area*

$$A_t \;:=\; \int_0^t X_s\,dY_s \;-\; \int_0^t Y_s\,dX_s\,.$$

a) Let $\alpha(t)$, $\beta(t)$ be C^1-functions, $p \in \mathbb{R}$, and

$$V_t \;=\; ipA_t \;-\; \frac{\alpha(t)}{2}\,\left(X_t^2 + Y_t^2\right) \;+\; \beta(t)\,.$$

Show using Itô's formula, that e^{V_t} is a local martingale provided $\alpha'(t) = \alpha(t)^2 - p^2$ and $\beta'(t) = \alpha(t)$.

b) Let $t_0 \in [0, \infty)$. The solutions of the ordinary differential equations for α and β with $\alpha(t_0) = \beta(t_0) = 0$ are

$$\begin{aligned} \alpha(t) &= p \cdot \tanh(p \cdot (t_0 - t))\,, \\ \beta(t) &= -\log\cosh(p \cdot (t_0 - t))\,. \end{aligned}$$

Conclude that

$$E\left[e^{ipA_{t_0}}\right] \;=\; \frac{1}{\cosh(pt_0)} \qquad \forall\, p \in \mathbb{R}\,.$$

c) Show that the distribution of A_t is absolutely continuous with density

$$f_{A_t}(x) \;=\; \frac{1}{2t\cosh(\frac{\pi x}{2t})}\,.$$