**Institut für angewandte Mathematik**
**Winter Semester 11/12**
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# "Stochastic Analysis", Problem sheet 8.

**Please hand in the solutions before Tuesday 6.12., 2 pm**

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**1. (Itô's Isometry and stochastic integrals with general previsible integrands)**
Let $(X_t)_{t\in[0,1]}$ be a square-integrable càdlàg martingale, and let $\mathcal{E}$ denote the set of elementary previsible process $(G_t)_{t\in[0,1]}$ of the form

$$G_t = \sum_{s\in\pi} A_s \, \mathbf{1}_{(s,s']} \,,$$

where $\pi$ is a partition of $[0,1]$, and the random variables $A_s$ are $\mathcal{F}_s$ measurable and bounded.

a) Prove directly that for any $G \in \mathcal{E}$,

$$E\left[\left(\int_0^1 G dX\right)^2\right] = E\left[\int_0^1 G^2 \, d[X]\right].$$

b) Let $\mathcal{H}^2$ denote the closure of $\mathcal{E}$ w.r.t. the $L^2$ norm

$$\|G\| \ := \ E\left[\int G^2 d[X]\right]^{1/2}$$

Give a definition of the integral $\int G \, dX$ and prove that it exists for $G \in \mathcal{H}^2$.

c) Prove that $\mathcal{H}^2$ contains all left-continuous bounded adapted processes $G$, and identify the integral for $G = H_-$ with $H$ bounded adapted and càdlàg with the stochastic integral defined in the course.

**2. (Concentration of measure)**

Let $M$ be a continuous local martingale satisfying $M_0 = 0$. Show that

$$P\left[\max_{s\le t} M_s \ge y \,,\, [M]_t \le K\right] \ \le \ \exp\left(-\frac{y^2}{2K}\right) \quad \forall\, t, y, K > 0 \,.$$

**3. (Exit distributions for Bessel and compund Poisson processes)**

a) Let $(X_t)_{0\le t<\zeta}$ be a solution of the **Bessel equation**

$$dX_t \;=\; -\frac{d-1}{2X_t}\,dt \;+\; dB_t, \qquad X_0 = x_0,$$

where $(B_t)_{t\ge 0}$ is a standard Brownian motion and $d$ is a real constant.

i) Find a non-constant function $u : \mathbb{R} \to \mathbb{R}$ such that $u(X_t)$ is a local martingale.

ii) Compute the ruin probability $P[T_a < T_b]$ for $a, b \in \mathbb{R}$ with $x_0 \in [a, b]$ .

iii) Proceeding similarly, determine the mean exit time $E[T]$, where $T = \min\{T_a, T_b\}$.

b) Now let $(X_t)_{t\ge 0}$ be a compound Poisson process with $X_0 = 0$ and jump intensity measure $\nu = N(m, 1),\ m > 0$.

i) Determine $\lambda \in \mathbb{R}$ such that $\exp(\lambda X_t)$ is a local martingale.

ii) Prove that for $a < 0$,

$$P[T_a < \infty] \;=\; \lim_{b\to\infty} P[T_a < T_b] \;\le\; \exp(ma/2).$$

Why is it not as easy as above to compute the ruin probability $P[T_a < T_b]$ exactly ?

**4. (Lévy's characterization)** Show via Lévy's theorem:

a) If $(B_t)$ is a Brownian motion and $c > 0$, then $\left(\sqrt{c}B_{t/c}\right)$ is also a Brownian motion.

b) If $(B^i_t)$ $(i = 1, \ldots, n)$ are independent Brownian motions, then

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} B^i_t \quad (t \ge 0)$$

is a Brownian motion.