Institut für angewandte Mathematik
Winter Semester 11/12
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"Stochastic Analysis", Problem sheet 7.

Please hand in the solutions before Tuesday 29.11., 2 pm

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1. (Drift and diffusion coefficient of Ito diffusions) Suppose that (B_t) is a d-dimensional Brownian motion, and $(X_t) = (X_t^1, \dots, X_t^d)$ is a solution of the SDE

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t, \qquad t \geq 0, \qquad X_0 = x \in \mathbb{R}^d,$$

with continuous bounded coefficients $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$. Determine the limits

$$\lim_{t\downarrow 0} \frac{1}{t} E\left[X_t^i - x^i\right], \qquad \lim_{t\downarrow 0} \frac{1}{t} E\left[(X_t^i - x^i)(X_t^j - x^j)\right], \qquad i,j = 1\dots,d.$$

2. (Hermite polynomials and the Kailath-Segal identity) Let M be a continuous local martingale such that $M_0 = 0$. We define the iterated stochastic integrals of M by

$$I_t^0 = 1, \quad I_t^n = \int_0^t I_s^{n-1}\,dM_s.$$

a) Prove that for $n \geq 2$,

$$nI_t^n = I_t^{n-1} M_t - I_t^{n-2} [M]_t \ .$$

b) Relate this identity to a recurrence formula for Hermite polynomials. Conclude that

$$n!\, I_t^n = H_n(M_t, [M]_t) \qquad \text{where} \qquad H_n(x,t) = t^{n/2} h_n(x/\sqrt{t}).$$

Hint: The Hermite polynomial h_n of order n is defined by the identity

$$\sum_{n\geq 0} \frac{u^n}{n!} h_n(x) = \exp\left(ux - \frac{u^2}{2}\right), \quad u, x \in \mathbb{R},$$

which implies

$$h_n(x) = (-1)^n \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n}\left(\exp\left(-\frac{x^2}{2}\right)\right), \qquad \textit{and}$$

$$\exp\left(ux - \frac{au^2}{2}\right) = \sum_{n\geq 0} \frac{u^n}{n!} H_n(x,a) \qquad \textit{for } a > 0.$$

3. (Exit times for Brownian motion with drift) Let

$$X_t := \; x + mt + \sigma B_t, \qquad t \geq 0,$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion, and $x, m \in \mathbb{R}, \sigma > 0$ are finite constants.

a) Let $a < x < b$. Compute the probability that (X_t) reaches the value b before it reaches a. *Hint: For* $m \neq 0$ *find* $\lambda \in \mathbb{R}$ *such that* $\exp(\lambda X_t)$ *is a martingale.*

b) If $m < 0$, show that $X^* := \sup_{t\geq 0} X_t$ is $\mathbb{P}$−a.s. finite. Furthermore, prove that $X^* - x$ is exponentially distributed with parameter $\alpha = 2|m|\sigma^{-2}$, i.e.,

$$\mathbb{P}[X^* \geq x + b] = \exp(-\alpha b) \;\text{ for any } b \geq 0.$$

c) In the case $m \geq 0$ prove that $T_b := \inf\{t \geq 0 | X_t = b\}$ is a.s. finite for any $b > x$, and compute the Laplace transform $E[\exp(-\lambda T_b)]$ for $\lambda \geq 0$.

4. (Localization by stopping) Let T be an $(\mathcal{F}_t)$ stopping time, G, H $(\mathcal{F}_t)$ adapted càdlàg processes, and X, Y $(\mathcal{F}_t)$ semimartingales.

a) Let (π_n) be a sequence of partitions of $\mathbb{R}^+$ such that $\lim_{n\to\infty} \text{mesh}(\pi_n) = 0$. Prove that the Riemann sum approximations $\sum_{s\in\pi_n} H_s(X_{s'\wedge t} - X_{s\wedge t})$ converge to $(H_- \cdot X)_t$ uniformly on compact intervals in probability.

b) Suppose that $G_t = H_t$ for $t < T$ and $X_t = Y_t$ for $t \leq T$. Prove that almost surely, $H_- \cdot X = G_- \cdot Y$ on $[0, T]$, and conclude that

$$(H_- \cdot X)^T = H_- \cdot X^T = (H_- \mathbf{1}_{[0,T]}) \cdot X\,.$$

c) Prove that $[X, Y]^T = [X^T, Y] = [X, Y^T] = [X^T, Y^T]$.