Institut für angewandte Mathematik
Winter Semester 11/12
Andreas Eberle, Evangelia Petrou

"Stochastic Analysis", Problem sheet 6.

Please hand in the solutions before Tuesday 22.11., 2 pm

---

1. (Approximating solutions of SDE - A counterexample) Let $(X_t)_{t\geq 0}$ be a continuous adapted stochastic process, and let (X_t^n) be the mollified process defined by

$$X_t^n = n\int_{(t-\frac{1}{n})^+}^t X_s\,ds \qquad \text{for } t \geq 0.$$

a) Show that for each $n \in \mathbb{N}$, X^n is a semimartingale, and solve the SDE

$$dZ_t^n = Z_t^n\,dX_t^n, \quad Z_0^n = 1.$$

b) Prove that for $t > 0$, $\lim_{n\to\infty} X_t^n = X_t$ a.s., but if X is a Brownian motion then Z_t^n does **not** converge to the unique solution of the limit SDE

$$dZ_t = Z_t\,dX_t, \quad Z_0 = 1.$$

2. (Exponential moments of exit times) Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and for the positive constants a, b define $T := \inf\{t > 0 : B_t \in (-a,b)^C\}$.

a) Show that for $\theta \in \mathbb{R}$,

$$X_t = \exp\left(\frac{1}{2}\theta^2 t\right)\cos\left[\theta\left(B_t - \frac{b-a}{2}\right)\right],$$

is a local martingale with

$$X_T = \exp\left(\frac{1}{2}\theta^2 T\right)\cos\left[\frac{b+a}{2}\theta\right].$$

b) For $0 \leq \theta \leq \frac{\pi}{a+b}$, show that $(X_{t\wedge T})_{t\geq 0}$ is a positive supermartingale, and deduce

$$\cos\left[\frac{b+a}{2}\theta\right] E\left[\exp\left(\frac{1}{2}\theta^2 T\right)\right] \leq \cos\left[\frac{b-a}{2}\theta\right].$$

c) Use b) to show that $X_T^* = \sup_{s\leq T}|X_s| \in \mathcal{L}^1$, and conclude that X is a martingale.

d) Conclude that

$$E\left[\exp\left(\frac{1}{2}\theta^2 T\right)\right] = \frac{\cos\left[\frac{b-a}{2}\theta\right]}{\cos\left[\frac{b+a}{2}\theta\right]}$$

3. (Naive stochastic integration is impossible) Let $A : [0,1] \to \mathbb{R}$ be a continuous function, and let (π_n) be a sequence of partitions of $[0,1]$ with $\lim_{n\to\infty} \operatorname{mesh}(\pi_n) = 0$. For any function $H : [0,1] \to \mathbb{R}$ we define

$$I_n(H) := \sum_{s\in\pi_n} H_s\big(A_{s'} - A_s\big),$$

Show that if $I_n(H)$ coverges for every continuous function H to a finite limit as $n \to \infty$, then A has bounded variation.

Hint: *Use the uniform boundedness principle (Banach-Steinhaus Theorem).*

4. (Stochastic integration and SDE w.r.t. Poisson point processes) Let (N_t) be a Poisson point process on a σ-finite measure space $(S, \mathcal{S}, \nu)$, and let $N(dt\,dy)$ denote the corresponding Poisson random measure on $[0,\infty) \times S$.

a) Define the stochastic integral $\int_{(0,t]\times S} H_s(y)\, N(ds\,dy)$ for an elementary predictable process $(t, y, \omega) \mapsto H_t(y)(\omega)$ of type

$$H_t \;=\; \sum_{i=1}^{n} A_i\, I_{(s_i,s_{i+1}]}(t)$$

with bounded $\mathcal{S} \otimes \mathcal{F}_{s_i}$ measurable random variables $(y, \omega) \mapsto A_i(y)(\omega)$. Prove that

$$E\left[\int_{(0,t]\times S} H_s(y)\, N(ds\,dy)\right] \;=\; E\left[\int_{(0,t]}\int_S H_s(y)\,\nu(dy)\,ds\right],$$

and show that the process

$$M_t^H \;:=\; \int_{(0,t]\times S} H_s(y)\, N(ds\,dy) \;-\; \int_{(0,t]}\int_S H_s(y)\,\nu(dy)\,ds$$

is a martingale.

b) Extend the definition of the stochastic integral and the properties above to processes that can be approximated by elementary predictable processes w.r.t. the $L^1([0,1] \times S \times \Omega, \lambda \otimes \nu \otimes P)$ norm. Use this definition to make sense of the SDE

$$dX_t \;=\; \int_S b(X_{t-}, y)\, N(dt\,dy),$$

where $b : \mathbb{R} \times S \to \mathbb{R}$ is a bounded function that is continuous in the first variable.

c) Give an intuitive description of a process (X_t) solving the SDE. Then prove formally that (X_t) solves the martingale problem for the operator

$$(\mathcal{L}f)(x) \;:=\; \int_S \big(f(x + b(x,y)) - f(x)\big)\,\nu(dy),$$

i..e., $f(X_t) - \int_0^t (\mathcal{L}f)(X_s)\,ds$ is a martingale for any $f \in C_b(\mathbb{R})$.