Institut für angewandte Mathematik
Winter Semester 11/12
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"Stochastic Analysis", Problem sheet 5.

Please hand in the solutions before Tuesday 15.11., 2 pm

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1. (Integration by Parts) Suppose that each of the processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ is the sum of a bounded martingale and an adapted càdlàg finite variation process on the filtered probability space $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$.

a) Define the stochastic integral $\int_0^t X_{s-} dY_s$, and prove the integration by parts formula

$$X_t Y_t - X_0 Y_0 = \int_0^t X_{s-} dY_s + \int_0^t Y_{s-} dX_s + [X,Y]_t \, .$$

Why is the covariation well-defined ?

b) Let Z be a bounded random variable and let A be a bounded continuous increasing process vanishing at 0. Prove that

$$E[ZA_\infty] \;=\; E\left[\int_0^\infty E[Z|\mathcal{F}_t]\, dA_t\right].$$

2. (Taylor expansion for exponentials of finite variation functions)
Let $A : [0,\infty) \to \mathbb{R}$ be a càdlàg function of finite variation.

a) Prove that if A is continuous then for any $k \in \mathbb{N}$,

$$\int_{(0,t]} \int_{(0,s_1)} \int_{(0,s_2)} \cdots \int_{(0,s_{k-1})} dA_{s_k}\, dA_{s_{k-1}} \cdots dA_{s_1} \;=\; (A_t - A_0)^k / k! \, .$$

Show that if A is increasing but not necessarily continuous then the right hand side still is an upper bound for the iterated integral.

b) Derive the expansion

$$\mathcal{E}_t^A \;=\; 1 + \sum_{k=1}^\infty \int_{(0,t]} \int_{(0,s_1)} \int_{(0,s_2)} \cdots \int_{(0,s_{k-1})} dA_{s_k}\, dA_{s_{k-1}} \cdots dA_{s_1}$$

for the exponential of A. Give an explicit expression for the remainder $R_t^{(n)}$ when the series expansion is truncated after n steps, and prove that

$$|R_t^{(n)}| \;\leq\; M_t\, V_t^{n+1}/(n+1)!$$

where $M_t = \sup_{s\leq t} |\mathcal{E}_s^A|$ and V_t is the variation of A on $[0,t]$.

3. (Change of measure for finite Markov chains) Let (X_t) on $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t))$ be a continuous time Markov chain with finite state space S and generator (Q-matrix) $\mathcal{L}$, i.e.,

$$M_t^f \;:=\; f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s)\,ds$$

is a martingale w.r.t. P for each function $f : S \to \mathbb{R}$. We assume $\mathcal{L}(a,b) > 0$ for $a \neq b$. Let

$$g(a,b) \;:=\; \tilde{\mathcal{L}}(a,b)/\mathcal{L}(a,b) - 1 \text{ for } a \neq b, \qquad g(a,a) \;:=\; 0,$$

where $\tilde{\mathcal{L}}$ is another Q-matrix.

a) To get started, verify once more on your own that

$$N_t \;:=\; \sum_{s \leq t} g(X_{s-}, X_s) \;-\; \int_0^t \sum_b \mathcal{L}(X_{s-}, b)\, g(X_{s-}, b)\, ds$$

is a martingale for any $g : S \times S \to \mathbb{R}$ with

$$N_t = \sum_{a,b:a\neq b} N_t^{a,b}, \qquad N_t^{a,b} = J_t^{a,b} - \mathcal{L}(a,b) L_t^a,$$

where $J_t^{a,b}$ denotes the number of jumps from a to b up to time t, and L_t^a is the time spent at a before t.

b) Let $\lambda(a) = \sum_{b\neq a} \mathcal{L}(a,b) = -\mathcal{L}(a,a)$ and $\tilde{\lambda}(a) = -\tilde{\mathcal{L}}(a,a)$ denote the total jump intensities at a. We define a "likelihood quotient" for the trajectories of Markov chains with generators $\tilde{\mathcal{L}}$ and $\mathcal{L}$ by $Z_t = \tilde{\zeta}_t / \zeta_t$ where

$$\tilde{\zeta}_t \;=\; \exp\left(- \int_0^t \tilde{\lambda}(X_s)\, ds \right) \prod_{s \leq t : X_{s-} \neq X_s} \tilde{\mathcal{L}}(X_{s-}, X_s),$$

and ζ_t is defined correspondingly. Prove that (Z_t) is the exponential of (N_t), and conclude that (Z_t) is a martingale with $E[Z_t] = 1$ for any t.

c) Let $\tilde{P}$ denote a probability measure on $\mathcal{A}$ that is absolutely continuous w.r.t. P on $\mathcal{F}_t$ with relative density Z_t for every $t \geq 0$. Show that for any $f : S \to \mathbb{R}$,

$$\tilde{M}_t^f \;:=\; f(X_t) - f(X_0) - \int_0^t (\tilde{\mathcal{L}}f)(X_s)\, ds$$

is a martingale w.r.t. $\tilde{P}$. Hence under the new probability measure $\tilde{P}$, (X_t) is a Markov chain with generator $\tilde{\mathcal{L}}$.

You may assume without proof that $(\tilde{M}_t^f)$ *is a local martingale w.r.t.* $\tilde{P}$ *if and only if* $(Z_t \tilde{M}_t^f)$ *is a local martingale w.r.t.* P.