Institut für angewandte Mathematik
Winter Semester 11/12
Andreas Eberle, Evangelia Petrou

"Stochastic Analysis", Problem sheet 4.

Please hand in the solutions before Tuesday 08.11., 2 pm

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1. (Geometric Poisson Processes and change of measure) Let $(N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$ on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t\geq 0}$.

a) Let $\sigma, \alpha \in \mathbb{R}$ with $\sigma > -1$. Solve the SDE

$$dS_t \;=\; \sigma\, S_{t-}\, dN_t \;+\; \alpha\, S_t\, dt\,, \qquad S_0 = 1,$$

by the ansatz $S_t = \exp(aN_t + bt)$. Given σ, for which value of α is (S_t) a martingale ?

b) Now let $\mu > 0$. Verify that

$$Z_t \;=\; (\mu/\lambda)^{N_t}\, e^{(\lambda-\mu)t}$$

is an $(\mathcal{F}_t)$ martingale with $E[Z_t] = 1$ for all t.
We define a new probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}_1)$ by

$$\tilde{\mathbb{P}}[A] \;=\; \int_A Z_1\, d\mathbb{P} \qquad \text{for any } A \in \mathcal{F}_1.$$

Verify that $\tilde{E}[X_t] = E[X_t Z_t]$ for any $\mathcal{F}_t$ measurable random variable X_t and $t \in [0,1]$. Compute the Laplace transforms $\tilde{E}[\exp(-uN_t)]$, $u \geq 0$, and the characteristic functions of the process $(N_t)_{t\in[0,1]}$ w.r.t. the new probability measure $\tilde{\mathbb{P}}$.

Conclude that under $\tilde{\mathbb{P}}$, $(N_t)_{t\in[0,1]}$ is a Poisson process with intensity μ.

2. (Subordinated Brownian motions) Let (B_t) be a one-dimesional Brownian motion on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with filtration $(\mathcal{F}_t)$.

a) Let (N_t) be a Poisson process with rate λ that is independent of the Brownian motion (B_t). Prove that $(\sigma B_{N_t})_{t\geq 0}$ is a compound Poisson process with Lévy measure

$$\nu(dx) = \frac{\lambda}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}\, dx\,.$$

b) More generally, let $(\tau_s)_{s\geq 0}$ be a subordinator with characteristic exponent ψ that is independent of the Brownian motion (B_t). Prove that $X_s = B_{\tau_s}$ is a Lèvy process with characteristic exponent

$$\psi_X(p) \;=\; \psi_\tau(p^2/2).$$

c) Let $\alpha \in (0,2)$. Show that $(\sqrt{2}B_t)$ subordinated by a stable process of index $\frac{\alpha}{2}$ is a symmetric stable process of index α.

Hint: *Use the fact that since (τ_s) is a non-negative valued process, via analytical extension, the equality $E[\exp(iu\tau_s)] = \exp(-\psi_\tau(u)s)$ holds for any $u \in \mathbb{C}$ with $Re(u) \geq 0$.*