Institut für angewandte Mathematik
Winter Semester 11/12
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"Stochastic Analysis", Problem sheet 3.

Please hand in the solutions before Wednesday 02.11., 11 am

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1. (Olber's paradox)
Suppose that stars occur in $\mathbb{R}^3$ at the points R_i, $i \in \mathbb{N}$, of a spatial Poisson process with intensity λ. The star at R_i has brightness B_i, where the B_i are i.i.d. with mean β. The total illumination at the origin from stars within a large ball with radius a is

$$I_a \;=\; \sum_{i:|R_i|\le a} \frac{cB_i}{|R_i|^2}$$

for some absolute constant c. Show that

$$E[I_a] \;=\; 4\pi\lambda c\beta a\,.$$

The fact that this is unbounded as $a \to \infty$ is called *Olber's paradox*, and suggests that the celestial sphere should be uniformly bright at night. The fact that it is not is a problem whose resolution is still a matter for debate. One plausible explanation relies on a sufficiently fast rate of expansion of the Universe.

2. (Chain rule for finite variation functions) Let $X : [0,\infty) \to \mathbb{R}$ be a càdlàg finite variation function, and let $f \in C^2(\mathbb{R})$. Prove that:

a) The function $t \mapsto f(X_t)$ has finite variation.

b) For every $t \in \mathbb{R}^+$,

$$f(X_t) - f(X_0) = \int_0^t f'(X_{s-})dX_s + \sum_{0<s\le t} \{f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\}.$$

3. (Simulation of Lévy processes) Implement on a computer:

a) Simulation of a symmetric α-stable process for a given parameter $\alpha \in (0,2)$.

b) Simulation of an α-stable subordinator for a given parameter $\alpha \in (0,1)$.

Discuss the limits of your approach and indicate possible improvements.

4. (Martingales of Poisson point processes with infinite intensity)
Let (N_t) be a Poisson point process on a $\sigma-$finite measure space $(S, \mathcal{S}, \nu)$. Prove the following statements by considering first elementary integrands:

a) For any $f \in \mathcal{L}^1(\nu)$ and $t \geq 0$,

1. $N_t^f = \int f(y) N_t(dy)$ is well defined and in $L^1(P)$;
2. $E[\int f(y) N_t(dy)] = t \int f(y) \nu(dy)$;
3. $\tilde{N}_t^f = N_t^f - t \int f(y) \nu(dy)$, $t \geq 0$, is a martingale.

b) For any $f, g \in \mathcal{L}^2(\nu) \cap \mathcal{L}^1(\nu)$ and $t \geq 0$,

1. $N_t^f, N_t^g \in L^2(P)$;
2. $\mathrm{Cov}[N_t^f, N_t^g] = t \int f(y) g(y) \nu(dy)$;
3. $\tilde{N}_t^f \tilde{N}_t^g - t \int f(y) g(y) \nu(dy)$, $t \geq 0$, is a martingale.

c) For any $f \in \mathcal{L}^1(\nu)$,

$$E\left[\exp(ipN_t^f)\right] \;=\; \exp\left(t \int \left(e^{ipf(y)} - 1\right) \nu(dy)\right).$$