Institut für angewandte Mathematik
Winter Semester 11/12
Andreas Eberle, Evangelia Petrou

"Stochastic Analysis", Problem sheet 2.

Please hand in the solutions before Tuesday 25.10., 2 pm

---

1. (Construction of Poisson point processes) Let $(S, \mathcal{S}, \nu)$ be a σ-finite measure space, and let $\lambda := \nu(S)$.

a) Suppose first that $\lambda \in (0, \infty)$. Prove that

$$N_t = \sum_{j=1}^{K_t} \delta_{\eta_j}, \qquad t \geq 0,$$

is a Poisson point process with intensity measure ν provided the random variables η_j, $j \in \mathbb{N}$, are independent with distribution $\lambda^{-1}\nu$, and (K_t) is an independent Poisson process of intensity λ.

b) Now consider the case $\lambda = \infty$. Let $(\nu_k)_{k\in\mathbb{N}}$ be a sequence of finite measures on $(S, \mathcal{S})$ with $\nu = \sum \nu_k$. Prove that if $(N_t^{(k)})_{t\geq 0}$, $k \in \mathbb{N}$, are independent Poisson point processes on $(S, \mathcal{S})$ with intensity measures ν_k then

$$\overline{N}_t = \sum_{k=1}^{\infty} N_t^{(k)}$$

is a Poisson point process with intensity measure ν.

2. (Quadratic variation and covariation) Let $(\pi_n)_{n\in\mathbb{N}}$ be sequence of partitions of $\mathbb{R}_+$ with $\mathrm{mesh}(\pi_n) \to 0$. The covariation of two functions $X, Y : [0, \infty) \to \mathbb{R}$ is defined as

$$[X, Y]_t \;=\; \lim_{n\to\infty} \sum_{s\in\pi_n} (X_{s'\wedge t} - X_{s\wedge t})(Y_{s'\wedge t} - Y_{s\wedge t}) \qquad \text{for any } t \geq 0$$

with $s' = \min\{u \in \pi_n : u > s\}$, provided the limit exists. The quadratic variation of X is $[X]_t = [X, X]_t$. Now suppose that (B_t) is a Brownian motion, and (N_t) is an independent Poisson process on a probability space $(\Omega, \mathcal{A}, P)$.

a) Prove that $[B]_t = t$ holds P-almost surely if $\sum_n \mathrm{mesh}(\pi_n) < \infty$.

b) Show that $[N]_t = N_t$.

c) Compute $[B, N]_t$.

3. (Càdlàg martingales) Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration on a probability space $(\Omega, \mathcal{A}, P)$.

a) Show that for a càdlàg $(\mathcal{F}_t)$ adapted process (X_t), the first hitting time

$$T_A \;=\; \inf\{t \geq 0 : X_t \in A\}$$

of a closed set $A \subset \mathbb{R}$ is **not** predictable in general.

b) Prove that a local $(\mathcal{F}_t)$ martingale (M_t) with càdlàg paths can be localised by a sequence $(M_{t\wedge T_n})$, $n \in \mathbb{N}$, of bounded martingales, provided the jumps of (M_t) are uniformly bounded, i.e., $\sup\{|\Delta M_t(\omega)| : t \geq 0, \omega \in \Omega\} < \infty$.

c) Give an example of a càdlàg local martingale that can not be localized by bounded martingales.