Institut für angewandte Mathematik
Winter Semester 11/12
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"Stochastic Analysis", Problem sheet 10.

Please hand in the solutions of Exercises 1-3 before Tuesday 10.1.2012, and those of Exercises 4 and 5 before Tuesday 17.1.2012, 2 pm.

MERRY CHRISTMAS AND A HAPPY NEW YEAR !

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1. (Brownian motion with drift) Let $b : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function and let (X_t) be a real valued stochastic process that satisfies the following equation:

$$dX_t = b(X_t)\, dt + dB_t, \quad X_0 = x.$$

a) Prove that for any $M, t \in (0, \infty)$ and $x \in \mathbb{R}$,

$$P[X_t \geq M] > 0.$$

b) Now choose $b(x) = -\alpha$ where α is a positive constant. Prove that

$$P\left[\lim_{t\to\infty} X_t = -\infty\right] \; = \; 1.$$

Compare this with the previous result.

2. (Time change I: Brownian motion on the unit sphere) Let $Y_t = B_t/|B_t|$ where $(B_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^n$, $n > 2$. Prove that the time-changed process

$$Z_a = Y_{T_a}, \qquad T = A^{-1} \text{ with } A_t = \int_0^t |B_s|^{-2}\, ds \; ,$$

is a diffusion taking values in the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ with generator

$$\mathcal{L}f(x) \; = \; \frac{1}{2}\left(\Delta f(x) - \sum_{i,j} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right) - \frac{n-1}{2} \sum_i x_i \frac{\partial f}{\partial x_i}(x), \qquad x \in S^{n-1}.$$

3. (Novikov' s condition)

a) Prove that a non-negative supermartingale (Z_t) satisfying $E[Z_t] = 1$ for any $t \geq 0$ is a martingale.

b) Now consider

$$Z_t = \exp\left(\int_0^t b(X_s) \cdot dX_s - \frac{1}{2}\int_0^t |b(X_s)|^2 \, ds\right),$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous vector field, and (X_t) is a Brownian motion w.r.t. the probability measure P.

(i) Show that (Z_t) is a supermartingale.

(ii) Prove that (Z_t) is a martingale if $|b(x)| \leq c \cdot (1+|x|)$ for some constant $c \in (0, \infty)$.

Hint: Prove first that $E[\exp \int_0^\varepsilon |b(X_s)|^2 \, ds] < \infty$ *for* $\varepsilon > 0$ *sufficiently small, and conclude that* $E[Z_\varepsilon] = 1$. *Then show by induction that* $E[Z_{k\varepsilon}] = 1$ *for any* $k \in \mathbb{N}$.

4. (Time change II: Lamperti's Theorem)

Prove that a geometric Brownian motion can be represented as a time-changed Bessel process:

$$\exp(B_t + \nu t) = R_{A_t},$$

where $A_t = \int_0^t \exp[2(B_s + \nu s)] \, ds$ and (R_t) is a Bessel process with parameter $d = 2\nu + 1$.

5. (Stochastic representation for solutions of PDE)

Let $f \in C_0^2(\mathbb{R}^n)$ and $\alpha(x) = (\alpha_1(x), \ldots, \alpha_n(x))$ with $\alpha_i \in C_0^2(\mathbb{R}^n)$ be given functions, and let $u \in C^2(\mathbb{R}_+ \times \mathbb{R}^n)$ be a bounded solution of the partial differential equation

$$\frac{\partial u}{\partial t} = \sum_{i=1}^n \alpha_i \frac{\partial u}{\partial x_i} + \frac{1}{2}\sum_{i=1}^n \frac{\partial^2 u}{\partial^2 x_i}, \qquad u(0, x) = f(x).$$

a) Show that

$$u(t, x) = E_x\left[\exp\left(\int_0^t \alpha(B_s) \cdot dB_s - \frac{1}{2}\int_0^t |\alpha(B_s)|^2 \, ds\right) f(B_t)\right],$$

where $(B_t)_{t \geq 0}$ is an $n-$dimensional Brownian motion starting at x w.r.t. P_x. In particular, there is at most one bounded solution.

b) Now assume that there exists a function $\gamma \in C_0^2(\mathbb{R}^n)$ such that $\alpha = \nabla\gamma$. Prove that

$$u(t, x) = \exp(-\gamma(x)) \, E_x\left[\exp\left\{-\frac{1}{2}\int_0^t \left[|\nabla\gamma|^2 + \Delta\gamma\right](B_s) ds\right\} \exp(\gamma(B_t)) f(B_t)\right].$$