Institut für angewandte Mathematik
Winter Semester 11/12
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"Stochastic Analysis", Problem sheet 1.

Please hand in the solutions before Tuesday 18.10., 2 pm

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1. (Brownian motion as a random Fourier series) Recall the Fourier series expansion of a continuous function on $[0, 1]$. By applying this to the paths of a Brownian motion B_t, prove the representation

$$B_t \;=\; Z_0 \cdot t + \sum_{k=1}^{\infty} Z_k \frac{\sqrt{2} \cdot \sin(k\pi t)}{k\pi}$$

with independent standard normal random variables $Z_0, Z_1, Z_2, \ldots$. In which sense does the series converge ? How can this representation be used to construct a Brownian motion ? Compare this construction to the Wiener-Lévy construction, and discuss advantages and disadvantages.

2. (Hitting times for Poisson processes) Let (N_t) be a Poisson process with intensity $\lambda > 0$ and $N_0 = 0$.

a) Prove that $M_t = (N_t - \lambda t)^2 - \lambda t$ is a martingale.

b) Let $T_a = \inf\{t \geq 0 : N_t = a\}$, where a is a positive integer. Show that

$$E[T_a] \;=\; \frac{a}{\lambda} \qquad \text{and} \qquad \mathrm{Var}\,[T_a] \;=\; \frac{a}{\lambda^2}.$$

3. (Martingales of compound Poisson processes) Consider a compound Poisson process given by

$$X_t \;=\; \sum_{i=1}^{N_t} Y_i, \qquad t \geq 0,$$

with a Poisson process (N_t) of intensity $\lambda > 0$ and independent i.i.d. random variables Y_i, $i \in \mathbb{N}$, with distribution ν, expectation value m and finite variance σ^2.

a) Prove that (X_t) is a Lévy process with symbol

$$\psi(p) \;=\; \int (1 - \exp(ip \cdot y))\, \lambda\, \nu(dy)\,.$$

b) Show that $M_t := X_t - m\lambda t$ is a martingale.

c) Suppose that ν is a normal distribution. For which values of λ is the process

$$Z_t \; = \; \exp(-X_t + (m - \frac{1}{2}\sigma^2)t)$$

a supermartingale ? Consider the two cases $m \leq -\sigma^2/2$ and $m \geq \sigma^2/2$.

4. (Properties of càdlàg functions)

a) Prove that if I is a compact interval, then for any càdlàg function $x : I \to \mathbb{R}$, the set $\{s \in I : |\Delta x_s| > \varepsilon\}$ is finite for any $\varepsilon > 0$. Conclude that any càdlàg function $x : [0, \infty) \to \mathbb{R}$ has at most countably many jumps.

b) Show that a uniform limit of a sequence of càdlàg functions is again càdlàg.