Institut für angewandte Mathematik
Wintersemester 10/11
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9. Übungsblatt „Grundzüge der Stoch. Analysis"

Abgabe bis Di 14.12., 14 Uhr, Postfach im Schließfachraum (LWK)

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1. (Itô isometry and stochastic integrals w.r.t. continuous martingales). Let $(M_t)_{t\geq 0}$ be a continuous L^2-bounded martingale w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$. Assume that there exists an adapted nondecreasing process $\langle M\rangle_t$ such that $M_t^2 - \langle M\rangle_t$ is a martingale.

a) Show that the Itô isometry

$$E\left[\left(\int_0^t Z_s\,dM_s\right)^2\right] = E\left[\int_0^t Z_s^2\,d\langle M\rangle_s\right]$$

holds for any predictable step function $(Z_t)_{t\geq 0}$.

b) Now suppose that the sample paths of the process $t \mapsto \langle M\rangle_t$ are almost surely absolutely continuous functions with bounded derivatives

$$d\langle M\rangle_t/dt \;\leq\; c\,, \qquad c \in (0,\infty).$$

Apply the Itô isometry to define the stochastic integral $\int_0^t H_s\,dM_s$ for any adapted process $H \in L^2(P\otimes\lambda)$.

2. (Stochastic integrals w.r.t. Itô processes). Let

$$I_s \;:=\; \int_0^s H_r\,dB_r, \qquad 0 \leq s \leq t,$$

with an $(\mathcal{F}_s)$–Brownian motion (B_s) on $(\Omega,\mathcal{A},P)$, and an $(\mathcal{F}_s)$–adapted process $H \in L^2(P\otimes\lambda)$. Suppose that (π_n) is a sequence of partitions of $[0,t]$ such that $|\pi_n| \to 0$. Prove that if $(G_s)_{0\leq s\leq t}$ is another $(\mathcal{F}_s)$–adapted continuous bounded process, then the Riemann sums $\sum_{s\in\pi_n} G_s\cdot(I_{s'} - I_s)$ converge in $L^2(P)$, and

$$\int_0^t G_s\,dI_s \;:=\; \lim_{n\to\infty}\sum_{s\in\pi_n} G_s\cdot(I_{s'} - I_s) \;=\; \int_0^t G_s\,H_s\,dB_s\,.$$

Hint: Express the Riemann sums as a stochastic integral $\int_0^t \ldots\, dB_s$ w.r.t. Brownian motion.

3. (A local martingale that is not a martingale). Let $(B_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}^3$ with initial value $B_0 = x$, $x \neq 0$. Show:

a) $X_t = 1/\|B_t\|$ is a local martingale up to $T = \infty$ w.r.t. the filtration $(\mathcal{F}_t)$ generated by (B_t).

b) $\{X_s |\, 0 \leq s \leq t\}$ is uniformly integrable for all $t \geq 0$.

c) X_t is *not* a martingale.

Hint: You may assume without proof the multi-dimensional Itô formula: If U is an open subset of $\mathbb{R}^d$ then for $F \in C^2(U)$,

$$F(B_t) - F(B_0) \;=\; \int_0^t \nabla F(B_s) \cdot dB_s + \frac{1}{2}\int_0^t \Delta F(B_s)\, ds \qquad \forall\, t < T_{U^C}$$

4. (Brownian motion writes your name). Prove that Brownian motion in $\mathbb{R}^2$ will write your name (in cursive script, without dotted i's or crossed t's).
To get the pen rolling, first take B_t to be a two-dimensional Brownian motion on $[0,1]$, and note that for any $[a,b] \subset [0,1]$ the process

$$X_t^{(a,b)} = (b-a)^{-1/2}(B_{a+t(b-a)} - B_a)$$

is again a Brownian motion on $[0,1]$. Now, take $g : [0,1] \to \mathbb{R}^2$ to be a parametrization of your name, and note that Brownian motion spells your name (to precision ϵ) on the interval $[a,b]$ if

$$\sup_{0\leq t\leq 1} |X_t^{a,b} - g(t)| \leq \epsilon. \tag{1}$$

a) Let A_k denote the event that inequality (1) holds for $a = 2^{-k-1}$ and $b = 2^{-k}$. Check that the events A_k are independent, and that one has $P[A_k] = P[A_1]$ for all k. Conclude that if $P[A_1] > 0$ then infinitely many of the A_k will occur with probability one.

b) Consider an extremely dull individual whose signature is maximally undistinguished so that $g(t) = (0,0)$ for all $t \in [0,1]$. This poor soul does not even make an X; his signature is just a dot. Show that

$$P\left[\sup_{0\leq t\leq 1} |B_t| \leq \epsilon\right] > 0. \tag{2}$$

c) Finally, complete the solution of the problem by using (2) and the Cameron-Martin Theorem to show that $P[A_1] > 0$; that is to prove

$$P\left[\sup_{0\leq t\leq 1} |B_t - g(t)| \leq \epsilon\right] > 0.$$