Institut für angewandte Mathematik
Wintersemester 10/11
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13. Übungsblatt „Grundzüge der Stoch. Analysis"

Abgabe bis Di 1.2., 14 Uhr, Postfach im Schließfachraum (LWK)

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1. (Covariation of Itô–processes and the Itô–Döblin–formula)

a) Compute the covariation of two Itô processes

$$I_t = \int_0^t G_s \, dB_s^1 \qquad \text{and} \qquad J_t = \int_0^t H_s \, dB_s^2 \,,$$

where B_t^1 and B_t^2 are independent $(\mathcal{F}_t)$–Brownian motions, and G_t and H_t are continuous $(\mathcal{F}_t)$–adapted processes.

b) State and prove a multi-dimensional Itô–Döblin–formula for the solution to the SDE

$$dX_t \;=\; b(X_t)dt + \sigma(X_t)dB_t \;,$$

where (B_t) is a d-dimensional Brownian motion and $b : \mathbb{R}^k \to \mathbb{R}^k$, $\sigma : \mathbb{R}^k \to \mathbb{R}^{k\times d}$ are continuous functions.

The results of Exercise 1 may be assumed for the following exercises.

2. (Stochastic oscillator)

a) Let A and σ be $d \times d$–matrices, and let B_t be a Brownian motion in $\mathbb{R}^d$. Show that the unique solution of the stochastic differential equation

$$dZ_t \;=\; AZ_t \, dt \;+\; \sigma \, dB_t \;, \qquad Z_0 = z_0,$$

is given by

$$Z_t \;=\; e^{tA} Z_0 \;+\; \int_0^t e^{(t-s)A} \sigma \, dB_s \;.$$

b) Small displacements from equilibrium (e.g. of a pendulum) with stochastic reset force are described by an SDE of type

$$\begin{aligned} dX_t &= V_t \, dt \\ dV_t &= -X_t \, dt \;+\; dB_t \end{aligned}$$

with a one-dimensional Brownian motion B_t
(in complex notation: $dZ_t = -iZ_t \, dt \;+\; i \, dB_t$, $Z_t = X_t + iV_t$).
Solve the SDE with initial condition $X_0 = x_0$, $V_0 = v_0$. Show that X_t is a normally distributed random variable with mean given by the solution of the corresponding deterministic equation. Compute the limit

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Var}[X_t] \;.$$

3. (Black-Scholes model) A stock price is modeled by a geometric Brownian Motion (S_t) with parameters $\alpha, \sigma > 0$. We assume that the interest rate is equal to a real constant r for all times. Let $c(t,x)$ be the value of an option at time t if the stock price at that time is $S_t = x$. Suppose that $c(t, S_t)$ is replicated by a hedging portfolio, i.e., there is a trading strategy holding ϕ_t shares of stock at time t and putting the remaining portfolio value $V_t - \phi_t S_t$ in the money market account with fixed interest rate r so that the total portfolio value V_t at each time t agrees with $c(t, S_t)$.
"Derive" the *Black-Scholes partial differential equation*

$$\frac{\partial c}{\partial t}(t,x) + rx\frac{\partial c}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 c}{\partial x^2}(t,x) \;=\; rc(t,x) \tag{1}$$

and the *delta-hedging rule*

$$\phi_t \;=\; \frac{\partial c}{\partial x}(t, S_t) \qquad (=: \text{ Delta }). \tag{2}$$

Hint: Consider the discounted portfolio value $\tilde{V}_t = e^{-rt}V_t$ and, correspondingly, $e^{-rt}c(t, S_t)$. Compute the Ito differentials, and conclude that both processes coincide if c is a solution to (1) and ϕ_t is given by (2).

4. (Cox-Ingersoll-Ross model) Let (B_t) be a Brownian motion. The Cox-Ingersoll-Ross model aims to describe for example an interest rate process (R_t) or a stochastic volatility process and is given by

$$dR_t \;=\; (\alpha - \beta R_t)dt + \sigma\sqrt{R_t}dB_t \;,$$

where α, β and σ are positive constants.

a) Show that $E[|R_t|^p] < \infty$ for all $t > 0$ and $p \geq 1$ by applying Itô's formula to $x \mapsto |x|^p$.

b) Compute the expectation of R_t. *Hint:* Apply Itô's formula to $f(t,x) = e^{\beta t}x$.

c) Proceed in a similar way to compute the variance of R_t. Find its asymptotic value

$$\lim_{t\to\infty} \mathrm{Var}[R_t] \;.$$

5. (Feynman and Kac at the stock exchange) The price of a security is modeled by geometric Brownian motion (X_t) with parameters $\alpha, \sigma > 0$. At a price x we have a cost $V(x)$ per unit of time. The total cost up to time t is then given by

$$A_t \;=\; \int_0^t V(X_s)ds \;.$$

Suppose that u is a bounded solution to the PDE

$$\frac{\partial u}{\partial t} \;=\; \mathcal{L}u \;-\; \beta V u \,, \qquad \text{where} \qquad \mathcal{L} \;=\; \frac{\sigma^2}{2}\,x^2\,\frac{d^2}{dx^2} \;+\; \alpha\, x\,\frac{d}{dx} \;.$$

Show that the Laplace transform of A_t is given by $E_x\left[e^{-\beta A_t}\right] = u(t,x)$.