Institut für angewandte Mathematik
Wintersemester 10/11
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12. Übungsblatt „Grundzüge der Stoch. Analysis"

Abgabe bis Di 25.1., 14 Uhr, Postfach im Schließfachraum (LWK)

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1. (Estimation of Real-world σ's). If X_t is a stochastic process that models the price of a security at time t, then the random variable

$$R_k(h) = \frac{X_{kh}}{X_{(k-1)h}} - 1$$

is called the *k th period return.* It expresses in percentage terms the profit that one makes by holding the security from time $(k-1)h$ to time kh. When we use geometric Brownian motion

$$dX_t = \mu X_t \, dt + \sigma X_t \, dB_t$$

as a model for the price of a security, one common misunderstanding is that σ can be interpreted as a normalized standard deviation of sample returns that more properly estimate

$$s = \sqrt{\mathrm{Var}[R_k(h)]/h} \; .$$

Sort out this confusion by calculating $E[R_k(h)]$ and $\mathrm{Var}[R_k(h)]$ in terms of μ and σ. Also, use these results to show that a honest formula for σ^2 is

$$\sigma^2 = \frac{1}{h} \log \left(1 + \frac{\mathrm{Var}[R_k(h)]}{(1 + E[R_k(h)])^2} \right),$$

and suggest how you might estimate σ^2 from the data $R_1(h), R_2(h), \ldots, R_n(h)$.

2. (Variation of constants). The technique used for solving Exercise 1 on sheet 11 can be applied to more general nonlinear stochastic differential equations of the form

$$dX_t = f(t, X_t) \, dt + c(t) X_t \, dB_t, \qquad X_0 = x,$$

where $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $c : \mathbb{R}^+ \to \mathbb{R}$ are continuous (deterministic) functions. Proceed as follows :

a) Find an explicit solution Z_t of the equation with $f \equiv 0$.

b) To solve the equation in the general case, use the Ansatz

$$X_t \; = \; C_t \cdot Z_t \; .$$

Show that the s.d.e. gets the form

$$\frac{dC_t(\omega)}{dt} \; = \; f(t, Z_t(\omega) \cdot C_t(\omega))/Z_t(\omega) \; ; \qquad C_0 = x. \tag{1}$$

Note that for each $\omega \in \Omega$, this is a *deterministic* differential equation for the function $t \mapsto C_t(\omega)$. We can therefore solve (1) with ω as a parameter to find $C_t(\omega)$.

c) Apply this method to solve the stochastic differential equation

$$dX_t = \frac{1}{X_t}\,dt + \alpha X_t\,dB_t\ ; \qquad X_0 = x > 0\ ,$$

where α is constant.

d) Apply the method to study the solution of the stochastic differential equation

$$dX_t = X_t^\gamma\,dt + \alpha X_t\,dB_t\ ; \qquad X_0 = x > 0\ ,$$

where α and γ are constants. For which values of γ do we get explosion?

3. (Lévy Area). If $c(t) = (x(t), y(t))$ is a smooth curve in $\mathbb{R}^2$ with $c(0) = 0$, then

$$A(t) \;=\; \int_0^t (x(s)y'(s) - y(s)x'(s))\,ds \;=\; \int_0^t x\,dy - \int_0^t y\,dx$$

describes the area that is covered by the secant from the origin to $c(s)$ in the interval $[0,t]$. Analogously, for a two-dimensional Brownian motion $B_t = (X_t, Y_t)$ with $B_0 = 0$, one defines the *Lévy Area*

$$A_t \;:=\; \int_0^t X_s\,dY_s \;-\; \int_0^t Y_s\,dX_s\,.$$

a) Let $\alpha(t)$, $\beta(t)$ be C^1-functions, $p \in \mathbb{R}$, and

$$V_t \;=\; ipA_t \;-\; \frac{\alpha(t)}{2}\left(X_t^2 + Y_t^2\right) \;+\; \beta(t)\,.$$

Show using Itô's formula, that e^{V_t} is a local martingale provided $\alpha'(t) = \alpha(t)^2 - p^2$ and $\beta'(t) = \alpha(t)$.

b) Let $t_0 \in [0,\infty)$. The solution of the ordinary differential equations for α and β with $\alpha(t_0) = \beta(t_0) = 0$ are

$$\begin{aligned}\alpha(t) &= p\cdot\tanh(p\cdot(t_0 - t))\,,\\ \beta(t) &= -\log\cosh(p\cdot(t_0 - t))\,.\end{aligned}$$

Conclude that

$$E\left[e^{ipA_{t_0}}\right] \;=\; \frac{1}{\cosh(pt_0)} \qquad \forall\, p \in \mathbb{R}\,.$$

c) Show that the distribution of A_t is absolutely continuous with density

$$f_{A_t}(x) \;=\; \frac{1}{2t\cosh(\frac{\pi x}{2t})}\,.$$

4. (Lévy's Arcsine law). State Lévy's Arcsine law for the time $A_t = \int_0^t I_{(0,\infty)}(B_s)ds$ spent by a standard Brownian motion (B_s) in the interval $(0,\infty)$. Prove it by proceeding in the following way :

*a) *(optional)* Let $\alpha, \beta > 0$. Show that if v is a bounded solution to the equation

$$\alpha v - \frac{1}{2}v'' + \beta I_{(0,\infty)} v \; = \; 1$$

on $\mathbb{R} \setminus \{0\}$ with $v \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ then

$$v(x) \; = \; E_x \left[\int_0^\infty \exp(-\alpha t - \beta A_t) dt \right] \qquad \text{for any } x \in \mathbb{R}.$$

b) Compute a corresponding solution v and conclude that

$$\int_0^\infty e^{-\alpha t} E_0 \left[e^{-\beta A_t} \right] dt \; = \; \frac{1}{\sqrt{\alpha(\alpha+\beta)}} \; .$$

c) Now use the uniqueness of the Laplace inversion to show that the distribution μ_t of A_t/t under P_0 is absolutely continuous with density

$$f_{A_t/t}(s) \; = \; \frac{1}{\pi\sqrt{s(1-s)}} \; .$$