Analysis on probability spaces and SPDE

3.4.12

1. Introduction

1.1. Stochastic dynamics in finite dimensions

$L \in \mathbb{R}^{n\times n}$ symmetric, e.g. $-L$ discrete Laplace operator

$V: \mathbb{R}^n \to [0,\infty)$

$U(x) = \frac{1}{2}\, x\cdot Lx + V(x) \quad , x \in \mathbb{R}^n$ ENERGY

$$\nabla U(x) = \underbrace{Lx}_{\text{linear part}} + \underbrace{\nabla V(x)}_{\text{non-linear perturbation}}$$

a) STATIC MODELS

(i) Deterministic: Equilibrium state $x = x_0$ is minimizer resp. critical point of U

$\rightsquigarrow$ variational problem $\nabla U(x_0) = 0$

(ii) Statistical mechanics: Assume L strictly positive definite.

Equilibrium distribution at temperature $T>0$ is

$$\mu_T(dx) = \frac{1}{Z_T} e^{-\frac{U(x)}{T}} dx \quad \text{BOLTZMANN DISTRIB.}$$

$$Z_T = \int_{\mathbb{R}^n} e^{-\frac{U(x)}{T}} dx \quad \text{normalization constant / partition function}$$

$T \to \infty$: all states equally likely

$T \in (0,\infty)$: states with low energy more likely

$T \to 0$: $\mu_T \xrightarrow{w} \delta_{x_0}$, x_0 global min. of U (if unique)

$$\mu_T(dx) = \frac{1}{\tilde{Z}_T} e^{-\frac{V(x)}{T}} N(0, TL^{-1})(dx) \text{ where}$$

$$N(0,TL^{-1})(dx) = \sqrt{\frac{\det L}{(2\pi T)^d}}\, e^{-\frac{x \cdot Lx}{2T}} dx$$

is Gaussian distribution with mean 0 and covariance matrix TL^{-1}

b) DYNAMIC MODELS

$\sigma \in \mathbb{R}^{n \times d}$, $a = \sigma\sigma^T \in \mathbb{R}^{d \times d}$

(more generally $\sigma = \sigma(x)$)

(i) Hamiltonian dynamics $x(t)$ position, $v(t)$ velocity at time $t \in \mathbb{R}$

$$\frac{dx}{dt} = v$$

$$\underbrace{m\frac{dv}{dt}}_{\text{derivative of momentum}} = \underbrace{-\gamma m v}_{\text{damping}} \underbrace{- a\nabla U(x)}_{\text{internal force}} + \underbrace{F(x)}_{\text{external force}}$$

$m > 0$ mass, $\gamma > 0$ damping, $F : \mathbb{R}^n \to \mathbb{R}^n$

Remark If a is invertible with $g = a^{-1}$ then

$$a\nabla U(x) = \operatorname{grad} U(x)$$

is the gradient of U w.r.t. metric

$$(v, w) := v \cdot g w \quad \left(= (\sigma^{-1} v) \cdot (\sigma^{-1} w) \right. \text{ if } \sigma \text{ is invertible}$$

(ii) Overdamped limit

$m \to 0$, $\gamma m = \text{const.}$, e.g. $= 1$

$$0 = -v - a\nabla U(x) + F(x)$$

$$\frac{dx}{dt} = -\underbrace{a\nabla U(x)}_{\text{grad}\, U(x)} + F(x) \qquad \text{GRADIENT FLOW}$$

Explicitly:

$$\frac{dx}{dt} = -aLx - a\nabla V(x) + F(x)$$

$\mathbb{R}^n \to$ function space , $-aL \to$ Laplace operator

$\rightsquigarrow$ semilinear parabolic PDE

AIM: Include noise (thermal fluctuations) in a way that is consistent with equilibrium model

(iii) Inertial Langevin dynamics

$T > 0$, assume L positive definite, $F \equiv 0$.

$(\Omega, \mathfrak{A}, P)$ probability space

$X_t : \Omega \to \mathbb{R}^n$, $V_t : \Omega \to \mathbb{R}^n$ stochastic processes solving the SDE

$$(*) \quad \begin{aligned} dX_t &= V_t \, dt \\ m \, dV_t &= -\gamma m V_t \, dt - \underbrace{\sigma \sigma^T \nabla U(X_t)}_{\substack{\operatorname{grad} U(X_t) \\ \text{intrinsic gradient}}} dt + \overbrace{\sqrt{2\gamma m T}}^{\substack{\text{random force due to} \\ \text{thermal fluctuations etc.}}} \underbrace{\sigma \, dW_t}_{\substack{\text{intrinsic} \\ \text{noise}}} \end{aligned}$$

where $W_t : \Omega \to \mathbb{R}^d$ is a <u>standard Brownian motion</u> (<u>Wiener process</u>), i.e., $W_0 = 0$ and for any $0 \le t_0 < t_1 < \dots < t_k$,

$$W_{t_i} - W_{t_{i-1}} \ (1 \le i \le k) \text{ independent } \sim N(0, (t_i - t_{i-1}) I_d)$$

Remarks 1) A solution of $(*)$ is a cont. stoch. process $(X_t, V_t)_{t\geq 0}$ on $(\Omega, \mathcal{A}, P)$ satisfying the integrated equations

$$X_t = X_0 + \int_0^t V_s \, ds$$

$$m V_t = m V_0 - \gamma m \int_0^t V_s \, ds - \int_0^t \sigma\sigma^T \nabla U(X_s)\, ds + \sqrt{2\gamma m T}\, \sigma W_t$$

P-almost surely for any $t \geq 0$.

2) If $U \in C^2(\mathbb{R}^n)$ then $\exists!$ solution (X_t, V_t) (up to modification on measure zero sets) for every initial condition $(x_0, v_0) \in \mathbb{R}^{2n}$, and (X_t, V_t) is a time-hom. Markov process on $\mathbb{R}^{2n}$ with generator

$$\mathcal{L} = v \cdot \nabla_x - \underbrace{\tfrac{1}{m} \nabla U(x) \cdot \sigma\sigma^T \nabla_v}_{(\operatorname{grad} U(x),\, \operatorname{grad}_v \,\cdot\,)} - \gamma\, v \cdot \nabla_v + \gamma m^{-1} T \underbrace{\nabla_v^T \sigma\sigma^T \nabla_v}_{\substack{\text{Laplacian} \\ \text{in } v\text{-component} \\ \text{w.r.t. } (\cdot,\cdot)}}$$

FACT The probability measure

$$\hat{\mu}_T(dx\,dv) = \frac{1}{\hat{Z}_T} e^{-\frac{1}{T}\left(\underbrace{\frac{m}{2}\overbrace{v\cdot a^{-1}v}^{=(v,v)=\|v\|^2}+U(x)}_{\text{Hamiltonian}}\right)} dx\,dv$$

on $\mathbb{R}^n\times\mathbb{R}^n$ is a stationary distribution for (X_t, V_t), i.e.,

$$(X_0, V_0) \sim \hat{\mu}_T \Rightarrow (X_t, V_t) \sim \hat{\mu}_T \quad \forall t \geq 0$$

$$\hat{\mu}_T = \mu_T \otimes N\left(0, \tfrac{T}{m} a\right)$$

Sketch of proof (informal):

$\nu_t := P\circ(X_t, V_t)^{-1}$ Distribution of (X_t, V_t) on $\mathbb{R}^n\times\mathbb{R}^n$

satisfies Fokker-Planck equation

$$\text{(FP)} \qquad \frac{d}{dt}\nu_t = \mathcal{L}^*\nu_t, \qquad \text{i.e.,}$$

$$\frac{d}{dt}\int f\,d\nu_t = \int \mathcal{L}f\,d\nu_t \quad \forall f \in C_0^\infty(\mathbb{R}^n)$$

Integration by parts shows that $\mathcal{L}^*\hat{\mu}_T = 0$, i.e.,

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} (\mathcal{L}f)(x,v)\,\hat{\mu}_T(dx\,dv) = 0 \quad \forall f \in C_0^\infty(\mathbb{R}^n).$$

Hence $\nu_t := \hat{\mu}_T$ is a stationary (constant) solution of (FP). The fact follows by uniqueness for (FP). $\square$

(iv) Overdamped Langevin dynamics

Overdamped limit of (*) $(m \to 0,\ \gamma m = 1)$:

$$(**) \quad dX_t = \underbrace{-\sigma\sigma^T \nabla U(X_t)}_{\text{grad } U(X_t)} dt + \sqrt{2T}\,\sigma\, dW_t$$

- SDE on $\mathbb{R}^n$ driven by d-dimensional Brownian motion $(W_t)_{t \geq 0}$
- Stochastic gradient flow w.r.t. metric $(v,w) = v \cdot a^{-1} w$:

$$(**') \quad dX_t = -\text{grad } U(X_t) + \sqrt{2T}\, dB_t$$

where $B_t := \sigma W_t$ is Brownian motion w.r.t. metric $(\cdot,\cdot)$

- Explicitly: $U(x) = \frac{1}{2} x \cdot Lx + V(x)$, $\sigma\sigma^T = a$

$$(**'') \quad dX_t = \underbrace{-aLX_t\, dt}_{\text{linear drift}} \underbrace{- a\nabla V(X_t) dt}_{\text{non-linear drift}} + \sqrt{2T}\,\sigma\, dW_t$$

$\mathbb{R}^n \to$ function space, $-aL \to$ Laplace operator

$\rightsquigarrow$ Semilinear parabolic SPDE

<u>Remarks</u> 1) If $U \in C^2(\mathbb{R}^n)$ then $\exists!$ solution $(X_t)_{t\geq 0}$ for a given initial condition $x_0 \in \mathbb{R}^n$. (X_t) is a <u>time-homogeneous Markov process</u> with generator

$$\mathcal{L} = T\Delta_g - (\operatorname{grad} U, \operatorname{grad} \cdot)$$

$$= T\nabla^T \sigma\sigma^T \nabla - b \cdot \nabla, \quad b = \sigma\sigma^T \nabla U$$

$$= T\sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i}$$

2) μ_T is a <u>stationary distribution</u> for (X_t).

<u>Sketch of proof</u>:

$(\mathcal{L}, C_b^\infty(\mathbb{R}^n))$ is a symmetric linear operator on $L^2(\mathbb{R}^n, \mu_T)$

$$\int f \mathcal{L}\tilde{f}\, d\mu_T = \frac{1}{Z_T}\int_{\mathbb{R}^n} f\,(T\overset{\text{div grad}}{\Delta_g}\tilde{f} - (\operatorname{grad} U, \operatorname{grad} \tilde{f}))\underbrace{e^{-U/T}}_{\to 0 \text{ as } |x|\to\infty} dx$$

$$\overset{\text{int. by parts}}{=} \frac{T}{Z_T}\int_{\mathbb{R}^n} (\operatorname{grad} f, \operatorname{grad}\tilde{f}) e^{-U/T} dx = T\underbrace{\int (\operatorname{grad} f, \operatorname{grad}\tilde{f})\, d\mu_T}_{\text{symmetric}}$$

for any $f \in C_b^\infty(\mathbb{R}^n)$

$$\overset{f=1}{\Longrightarrow} \int \mathcal{L}\tilde{f}\, d\mu_T = 0 \quad \forall \tilde{f} \in C_b^\infty(\mathbb{R}^n)$$

$$\Longrightarrow \mathcal{L}^* \mu_T = 0 \underset{\substack{\text{Fokker-Planck} \\ \text{+ uniqueness}}}{\Longrightarrow} \mu_T \text{ stationary distrib.}$$

$\square$

(v) The linear case $V \equiv 0$, $U(x) = \frac{1}{2} x \cdot Lx$

Overdamped Langevin equation:

$$(***) \quad dX_t = -aLX_t\, dt + \sqrt{2T}\, \sigma\, dW_t$$

- linear SDE with additive noise (i.e. σ does not depend on X_t)
- stationary distribution is Gaussian:

$$\mu_T = N(0, TL^{-1})$$

- solution of $(***)$ is a Gaussian process, i.e., $(X_{t_1}, X_{t_2}, \dots, X_{t_k})$ has a Gaussian distrib. $\forall t_1, \dots, t_k \geq 0$.

Explicit solution by variation of constants:

$\underline{T=0}$: $\quad X_t = e^{-taL} C, \quad C: \Omega \to \mathbb{R}^n$

where
$$e^{-taL} := \sum_{n=0}^{\infty} \frac{t^n}{n!} (aL)^n \quad \text{matrix exponential}$$

Remark $L \in \mathbb{R}^{n\times n}$ symmetric, $a = \sigma\sigma^T \in \mathbb{R}^{n\times n}$ symmetric, non-deg

$\Rightarrow$ aL is symmetric w.r.t. inner product $(v,w) = v \cdot a^{-1} w$:

$$(aLv, w) = (aLv)\cdot a^{-1} w = (Lv)\cdot w = v \cdot Lw = (v, aLw)$$

Spectral Theorem
$\Rightarrow$ $\exists$ orthonormal basis $\{e_1, \dots, e_n\}$ of $(\mathbb{R}^n, (\cdot,\cdot))$ consisting of eigenvectors of aL, and

$$aLv = \sum_{i=1}^{n} \lambda_i (v, e_i) e_i \quad \forall v \in \mathbb{R}^n$$

$$\Rightarrow \quad e^{-taL} v = \sum_{i=1}^{n} e^{-t\lambda_i} (v, e_i) e_i$$

$\underline{T \neq 0}$: Ansatz $X_t = \overbrace{e^{-taL}}^{\text{smooth}} \overbrace{C_t}^{\text{only continuous, not of bounded variation}}$, C_t stoch. process

$$\longrightarrow dX_t = -aLX_t\,dt + \underbrace{e^{-taL}\,dC_t}_{\overset{!}{=}\sqrt{2T}\,\sigma\, dW_t} \quad \text{(by Stieltjes calculus)}$$

$$dC_t \overset{!}{=} \sqrt{2T}\, e^{taL}\sigma\, dW_t$$

$$C_t = C_0 + \sqrt{2T} \int_0^t e^{raL}\sigma\, dW_r$$

$$(****) \quad X_t = e^{-taL} C_0 + \sqrt{2T}\int_0^t e^{(r-t)aL}\sigma\, dW_r$$

Remark 1) The integral is an Itô stochastic integral. In this case, it can be defined easily by integration by parts:

$$\int_0^t h_r\, dW_r := h_t W_t - h_0 W_0 - \int_0^t h'_r W_r\, dr$$

for any $h \in C^1([0,t], \mathbb{R}^{n\times n})$

2) $\int_0^t h_r\, dW_r$ is a Gaussian random variable with mean 0 and covariance matrix $C = \int_0^t h_r h_r^T\, dr$ (Exercise, cf. below).

THEOREM The unique solution of $(***)$ with initial condition $X_0 = C_0$ is given by $(****)$

Proof Exercise

(vi) The nonlinear case $V \neq 0$, $b(x) := a \cdot \nabla V(x)$

$$(**) \quad dX_t = -aLX_t\,dt \underbrace{- b(X_t)\,dt + \sqrt{2T}\,\sigma\,dW_t}$$

try again variation of constants : $X_t = e^{-taL} C_t$

$$dX_t = -aLX_t\,dt + \underbrace{e^{-taL}\,dC_t}$$

$$(**) \Leftrightarrow C_t = C_0 + \int_0^t e^{raL}\, b(X_r)\,dr + \sqrt{2T} \int_0^t e^{raL}\,\sigma\,dW_r \Leftrightarrow$$

$$(*****) \quad X_t = X_0 + \int_0^t e^{(r-t)aL}\, b(X_r)\,dr + \sqrt{2T} \int_0^t e^{(r-t)aL}\,\sigma\,dW_r$$

DEFINITION $(X_t)_{t\geq 0}$ is called a mild solution of the SDE (**) iff (*****) holds for any $t \geq 0$, P-almost surely.

We have shown:

THEOREM In the finite-dimensional case, (X_t) is a mild solution of the SDE (**) if and only if it is a strong solution.

On infinite dimensional spaces this equivalence is not true:

Example: $\mathbb{R}^n \rightsquigarrow L^2((0,1), \mathbb{R})$, $\sigma = I$, $V \equiv 0$,
$-L \rightsquigarrow \frac{d^2}{ds^2}$ with Dirichlet boundary conditions

Then the stoch. integral in (*****) is a well-defined process with values in $L^2(0,1)$ (even in $C([0,1])$), cf. below.
Hence for $V \equiv 0$ there is a mild solution $X_t : \Omega \to L^2(0,1)$, $t \geq 0$.
However, for a given $t \geq 0$ and a.e. $\omega \in \Omega$, the function $X_t(\omega) \in L^2(0,1)$ is ~~almost surely~~ not differentiable. Thus (X_t) is not a strong solution of (**).

1.2. A dynamic model for a random string (Funaki 1983)

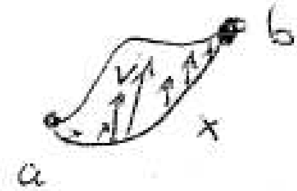
$a, b \in \mathbb{R}^d$ Elastic string

$x : [0,1] \to \mathbb{R}^d$ continuous, $x(0)=a$, $x(1)=b$

$$U(x) = \underbrace{\frac{1}{2}\int_0^1 |x'(s)|^2 \, ds}_{\text{elastic energy}} + \int_0^1 \phi(x(s))\, ds$$

$\phi : \mathbb{R}^d \to [0,\infty)$ potential, smooth

Derivatives of U: $v \in C_0^2((0,1), \mathbb{R}^d)$



$$\frac{\partial U}{\partial v}(x) = \frac{d}{d\varepsilon} U(x+\varepsilon v)\Big|_{\varepsilon=0}$$

$$= \int_0^1 v'(s)\cdot x'(s)\, ds + \int_0^1 v(s)\cdot \nabla\phi(x(s))\, ds$$

$$= \int_0^1 v(s)\cdot \left[-x''(s) + \nabla\phi(x(s))\right] ds$$

Gradient of U w.r.t. $L^2(0,1)$ metric:

$$\nabla_{L^2} U(x) = -x'' + (\nabla\phi)\circ x$$ "functional derivative" $\delta U/\delta x$

a) DETERMINISTIC DYNAMICS

$x(t,s)$, $v(t,s)$ position & velocity at time t and position s

$$\frac{\partial x}{\partial t} = v$$

$$m \frac{\partial v}{\partial t} = -\gamma m v - \nabla_{L^2} U(x)$$

Overdamped limit:

$$\frac{\partial x}{\partial t} = -\nabla_{L^2} U(x) = \frac{\partial^2 x}{\partial s^2} - (\nabla \phi) \circ x$$

$$\frac{\partial x}{\partial t}(t,s) = \frac{\partial^2 x}{\partial s^2}(t,s) - \nabla \phi\left(x(t,s)\right)$$

Semilinear pde

b) STOCHASTIC DYNAMICS WITH SPATIAL DISCRETIZATION

Fix $n \in \mathbb{N}$, $x^i(t) = x\left(t, \frac{i}{n}\right)$ $(i = 0, 1, \ldots, n)$

$$x(t,s) \longrightarrow x(t) = \left(x^i(t)\right)_{i=0}^{n}$$

State space : $E_n = \{ x = (x^0, \dots, x^n) : x^i \in \mathbb{R}^d,\ x^0 = a,\ x^n = b \} \cong \mathbb{R}^{d(n-1)}$

Energy : $$U_n(x) = \frac{1}{2} \sum_{i=0}^{n-1} \left| \frac{x^{i+1} - x^i}{1/n} \right|^2 \frac{1}{n} + \sum_{i=1}^{n} \phi(x^i)\, \frac{1}{n}$$

$$= \frac{n}{2} \sum_{i=0}^{n-1} |x^{i+1} - x^i|^2 + \frac{1}{n} \sum_{i=1}^{n} \phi(x^i)$$

Remark $$U_n(x) = \frac{1}{2}\, x \cdot L_n x + V_n(x)$$

where

$$(L_n x)^i = -n \left((x^{i+1} - x^i) - (x^i - x^{i-1}) \right) \quad \forall\, i = 1, \dots, n-1$$

i.e. $$L_n = -n\, \Delta_n \quad , \quad \Delta_n \text{ second difference operator}$$

$$V_n(x) = \frac{1}{n} \sum_{i=1}^{n} \phi(x^i)$$

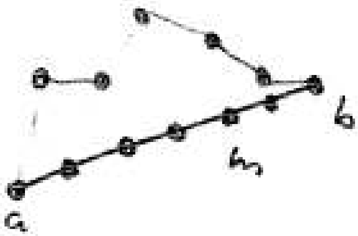
Equilibrium measure :

$$\mu^{\phi}_{T,n}(dx) = \frac{1}{Z_{T,n}}\, e^{-\frac{1}{nT} \sum_{i=1}^{n} \phi(x^i)}\, e^{-\frac{n}{2T} \sum_{i=0}^{n-1} |x^{i+1} - x^i|^2}\, \delta_a(dx^0) \prod_{i=1}^{n-1} dx^i\, \delta_b(dx^n)$$

$\phi \equiv 0$: Distribution of a Random Walk with increments $X^{i+1} - X^i \sim N(0, \frac{T}{n})$ conditioned to start at a and end at b after n steps.

<u>Remark</u> For $\phi \equiv 0$, $\mu^{\phi}_{T,n}$ is a Gaussian measure on E_n with mean vector

$$m^i = a + \frac{i}{n}(b-a)$$



and covariance matrix

$$C = \frac{T}{n}(-\Delta_n)^{-1} \quad \text{Green function of discrete Laplacian}$$

<u>Metric:</u> Tangent space $TE_n = \{v = (v^0, \dots, v^n) : v^i \in \mathbb{R}^d, v^0 = v^n = 0\}$

$$(v,w)_n = \frac{1}{n}\sum_{i=1}^{n-1} v^i \cdot w^i \quad \text{discrete } L^2 \text{ metric}$$

$$= v \cdot a^{-1} w \quad \text{where } a = nI, \ a = \sigma\sigma^T, \ \sigma = \sqrt{n}\, I$$

<u>Overdamped Langevin dynamics w.r.t. U_n and $(\cdot,\cdot)_n$:</u>

$$dX_t = -\operatorname{grad}_n U_n(X_t)\,dt + \sqrt{2T}\,\underbrace{dB_t}_{\text{Brownian motion w.r.t. } (\cdot,\cdot)_n}$$

$$= -n \nabla U_n(X_t)\,dt + \sqrt{2Tn}\,\underbrace{dW_t}_{\text{standard Brownian motion on } E}$$

$$(*)\quad \begin{cases} dX_t^i = -n \dfrac{\partial U_n}{\partial x^i}(X_t)\,dt + \sqrt{2T_n}\,dW_t^i & (i=1,\dots,n-1) \\ X_t^0 = a, \quad X_t^n = b \end{cases}$$

$$\frac{\partial U_n}{\partial x^i}(x) = -n\,\underbrace{\left(x^{i+1} - 2x^i + x^{i-1}\right)}_{\text{discrete Laplacian, second difference operator}} + \frac{1}{n}\nabla\phi(x^i)$$

$\longrightarrow$ Coupled system of SDEs in $\mathbb{R}^d$

$$(*')\quad \begin{cases} dX_t^i = n^2 \cdot \left(X_t^{i+1} - 2X_t^i + X_t^{i-1}\right) - \nabla\phi(X_t^i) + \sqrt{2T_n}\,dW_t^i \\ X_t^0 = a, \quad X_t^n = b \end{cases}$$

$W_t^1, \dots, W_t^{n-1} : \Omega \to \mathbb{R}^d$ independent Brownian motions

<u>Exercise</u> Simulate and visualize the dynamics defined by $(*')$ in the case $d=2$, $\phi \equiv 0$

C) THE CONTINUUM LIMIT

FOR THE EQUILIBRIUM MEASURE

Identify: $(x^i) \in E_n \longleftrightarrow$ piecewise linear function $x(s) = \begin{cases} x^i & \text{for } s = i/n \\ \text{linear inbetween} \end{cases}$

$$E_n \subset C([0,1], \mathbb{R}^d) =: E$$

$\mu^{\phi}_{T,n} \;\widehat{=}\;$ probability measure on E

$(X^{(n)}_t) \;\widehat{=}\;$ stochastic process on E

Limit as $n \to \infty$?

Equilibrium measure for $\phi \equiv 0$

$$\mu^0_{T,n}(dx) = \frac{1}{Z_{T,n}} \exp\left(-\frac{n}{2T}\sum_{i=0}^{n-1} |x^{i+1} - x^i|^2\right) \delta_a(dx^0) \prod_{i=1}^{n-1} dx^i \, \delta_b(dx^n)$$

- This is the marginal distribution of a Brownian bridge from a to b:

$x(s) := a + \sqrt{T}\,\beta(s)$, β standard Brownian motion

$\Rightarrow x(0) = a$, $x\left(\frac{i+1}{n}\right) - x\left(\frac{i}{n}\right)$ independent $\sim N\left(0, \frac{T}{n} I\right)$

$\Rightarrow \left(x(0), x\left(\frac{1}{n}\right), \ldots, x(1)\right) \sim \frac{1}{\tilde{Z}_{T,n}} \exp\left(-\frac{n}{2T}\sum_{i=0}^{n-1} |x^{i+1} - x^i|^2\right) \delta_a(dx^0) \prod_{i=1}^{n} dx^i$

$\Rightarrow \left(x(0), x\left(\frac{1}{n}\right), \ldots, x(1)\right) \,\big|\, x(1) = b \sim \mu^0_{T,n}$

<u>DEFINITION</u> A probability measure μ_T^0 on the Borel σ-algebra $\mathcal{B}(E)$ is called <u>pinned Wiener measure from a to b with diffusion coefficient $T>0$</u> iff for any $k\in\mathbb{N}$ and $0=s_0<s_1<\dots<s_k=1$,

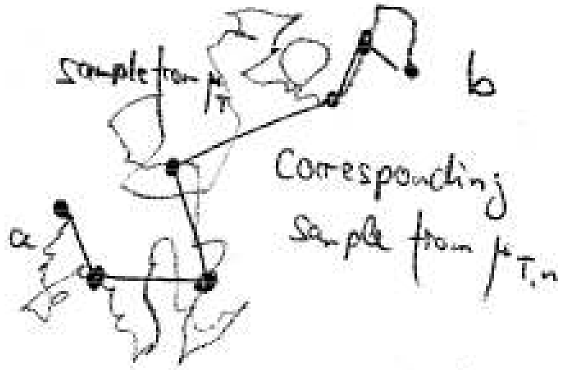
$$\mu_T^0\left(\{x\in E : (x(s_0),\dots,x(s_k))\in A\}\right)$$
$$\propto \int_A \exp\left(-\frac{1}{2T}\sum_{i=0}^{k-1}\frac{|x^{i+1}-x^i|^2}{s_{i+1}-s_i}\right)\delta_a(dx^0)\prod_{i=1}^{k-1}dx^i\,\delta_b(dx^k)$$

<u>Remark</u> 1) $\mathcal{B}(E)=\sigma\left(x\mapsto x(s) : s\in[0,1]\right)$

2) $\exists!$ pinned Wiener measure μ_T^0 on E for given parameters a,b,T, cf. below. Moreover, μ_T^0-almost every function $x\in E$ is α-Hölder continuous for any $\alpha<1/2$ but not for $\alpha\geq 1/2$.

<u>THEOREM</u> 1) $\mu_{T,n}^0$ is the distribution of $\left(x(\tfrac{i}{n})\right)_{i=0}^{n}$ w.r.t. pinned Wiener measure μ_T^0. Hence $\mu_{T,n}^0$ corresponds to the measure on piecewise linear functions obtained as the image of μ_T under piecewise linear interpolation.

2) $\mu_{T,n}^0 \xrightarrow{w} \mu_T^0$, i.e., $\int F\,d\mu_{T,n}^0 \to \int F\,d\mu_T^0 \;\forall F\in C_b(E)$.



Proof: Exercise

3) $\mu^{\phi}_{T,n} \xrightarrow{w} \mu^{\phi}_{T}$ where $\mu^{\phi}_{T}(dx) = \frac{1}{Z^{\phi}_{T}} e^{\int_0^1 \phi(x(s))\,ds} \mu^{0}_{T}(dx)$.

Remarks 1) Informally,

$$\text{"}\mu_T(dx) \propto \exp\left(-\frac{1}{2T}\int_0^1 |x'(s)|^2 ds\right) \delta_a(dx(0)) \prod_{s\in(0,1)} dx(s)\, \delta_b(dx(1))\text{"}$$

Does not make sense rigorously since $\int_0^1 |x'(s)|^2 ds = \infty$ almost surely.

2) Rigorously, μ^0_T is a Gaussian measure on the Banach space E with mean $m(s) = a + s(b-a)$, i.e.,

$$E_{\mu^0_T}\left[(f,\cdot)_{L^2(0,1)}\right] = (f,m)_{L^2(0,1)} \quad \forall f \in L^2([0,1],\mathbb{R}^d),$$

and covariance

$$\mathrm{Cov}_{\mu^0_T}\left[(f,\cdot)_{L^2(0,1)},(g,\cdot)_{L^2(0,1)}\right] = T\,(f,(-\Delta)^{-1}g)_{L^2(0,1)} \quad \forall f,g \in L^2([0,1],\mathbb{R}^d)$$

where Δ denotes the self-adjoint realization of the operator $\frac{d^2}{ds^2}$ with Dirichlet boundary conditions on $L^2([0,1],\mathbb{R}^d)$, see Section 2.

3) Explicitly, the "covariance operator" $G := (-\Delta)^{-1}$ is given by

$$(Gf)(s) = \int_0^1 G(s,u)\, f(u)\, du \qquad \forall f \in L^2([0,1], \mathbb{R}^d)$$

where

$$G(s,u) = \begin{cases} s\cdot(1-u) & \text{for } s \le u \\ (1-s)\cdot u & \text{for } s \ge u \end{cases} = s \wedge u - su$$

is the corresponding Green's function. In particular:

$$\mathrm{Cov}_{\mu_T^0}\left(x^k(s), x^\ell(u)\right) = T \cdot \delta_{k\ell} \cdot G(s,u) \quad \forall k,\ell \in \{1,\dots,d\}, s,u \in [0,1]$$

d) CONTINUUM LIMIT FOR STOCHASTIC DYNAMICS

Stochastic dynamics for finite-dimensional model:

$$dX_t = -\mathrm{grad}_n\, U_n(X_t)\, dt + \sqrt{2T}\, dB_t$$

grad_n = Gradient w.r.t. discrete L^2 metric $(v,w)_n = \frac{1}{n}\sum_{i=1}^n v^i \cdot w^i$

B = Brownian motion w.r.t. discrete L^2 metric

(= $\sqrt{n}$ × Standard Brownian motion w.r.t. Euclidean metric)

$$(v, B_t - B_s)_n \sim N\left(0, (t-s)\|v\|_n^2\right) \quad \forall v \in TE_n,\ 0 \le s \le t$$

$$\overset{\shortparallel}{\underbrace{\tfrac{1}{n}\, v \cdot (B_t - B_s)}_{\sim N(0,(t-s)n|v|^2)}} \qquad \sim N\left(0, \overset{\shortparallel}{\tfrac{t-s}{n}} |v|^2\right)$$

Infinite-dimensional model: Formally

$$\text{“}\quad dX_t = -\nabla_{L^2} U(x) + \sqrt{2T}\, dB_t \quad\text{”}$$

where

$$\nabla_{L^2} U(x) = \frac{\delta U}{\delta x} = -x'' + (\nabla\phi)\circ x,$$

$$B = \text{Brownian motion w.r.t. } L^2(0,1) \text{ metric.}$$

However:

1) A Brownian motion w.r.t. the $L^2(0,1)$ metric does not exist as a process with values in the Hilbert space $L^2(0,1)$.

2) Even after making sense of (B_t), the process (X_t) does not take values in a space of differentiable functions.

Way out:

1) Replace $L^2(0,1)$ by a larger space $\hat{E}$ (consisting of distributions instead of functions).

2) Consider mild instead of strong solutions.

$\hat{E}$ Banach space such that

$L^2(0,1) := L^2([0,1], \mathbb{R}^d) \subseteq \hat{E}$ densely and continuously, i.e.,

(*) $\|v\|_{L^2(0,1)} \leq \text{const.} \|v\|_{\hat{E}}$, $L^2(0,1)$ dense in $\hat{E}$.

(E.g. $\hat{E} = H^{-\alpha}$ Sobolev space of negative order, cf. below.)

$\hat{E}^* :=$ dual space $=$ all continuous linear functions $\ell : \hat{E} \to \mathbb{R}$

$\ell \in \hat{E}^* \overset{(*)}{\Rightarrow} \ell : L^2(0,1) \to \mathbb{R}$ continuous w.r.t. L^2 norm

$\Rightarrow \ell \in L^2(0,1)^*$

$\overset{\text{Riesz}}{\Longrightarrow} \exists!\, v_\ell \in L^2(0,1) : \forall w \in L^2(0,1) \quad \ell(w) = (v_\ell, w)_{L^2(0,1)}$

$\hat{E}^* \subseteq L^2(0,1)^* \overset{\text{Riesz}}{\cong} L^2(0,1) \subseteq \hat{E}$

<u>Definition</u> A stochastic process $B_t : \Omega \to \hat{E}$ is called a <u>Wiener process w.r.t. the $L^2(0,1)$ metric</u> iff it has independent increments over disjoint time intervals, and

$$\ell(B_t - B_s) \sim N\left(0, (t-s)\|v_\ell\|^2_{L^2(0,1)}\right) \quad \forall \ell \in \hat{E}', \ 0 \le s < t.$$

<u>Rem.</u> 1) $(B_t)_{t\geq 0}$ is $L^2(0,1)$ Wiener process

$$\iff \forall \ell \in \hat{E}' : (\ell(B_t))_{t\geq 0} \text{ is BM with variance } \|v_\ell\|^2_{L^2(0,1)}$$

2) An $L^2(0,1)$ Wiener process does <u>not</u> exist on $\hat{E} = L^2([0,1], \mathbb{R}^d)$. However, it exists on any Hilbert space $\hat{E} \supset L^2([0,1], \mathbb{R}^d)$ s.t. the embedding of $L^2([0,1], \mathbb{R}^d)$ into $\hat{E}$ is Hilbert-Schmidt, cf. below

<u>b) Mild solution of (*)</u> In analogy to Section 1.1 we define:

<u>Def.</u> $(X_t)_{t\geq 0}$ is called a <u>mild solution</u> of (*) iff

$$X_t = X_0 + \int_0^t e^{(t-s)\Delta} F(X_s)\, ds + \sqrt{2T} \int_0^t e^{(t-s)\Delta}\, dB_s \quad \forall t \geq 0.$$

Here $e^{t\Delta}$ denotes the semigroup generated by the self-adjoint realization of the Laplacian with Dirichlet boundary conditions on $L^2(0,1)$. Due to the regularizing effect of $e^{t\Delta}$ it can be shown that the integrals are well-defined and take values in $C([0,1], \mathbb{R}^d)$

2. Gaussian Measures

Ref. V.I. Bogachev: Gaussian Measures, AMS

2.1. Gaussian measures on $\mathbb{R}^1$

<u>Def.</u> A probability measure μ on $\mathcal{B}(\mathbb{R}^1)$ is called <u>Gaussian</u> if

$\mu = \delta_m$ for some $m \in \mathbb{R}$ or

$$\mu(dx) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{|x-m|^2}{2v}\right) dx$$

for some $m \in \mathbb{R}$ and $v > 0$.

$m = \int x \, \mu(dx)$ mean, $\quad v = \int (x-m)^2 \mu(dx) \in [0,\infty)$ variance

<u>Fact</u> μ Gaussian with mean $m \in \mathbb{R}$ and variance $v \in [0,\infty)$

$$\Longleftrightarrow \quad \varphi_\mu(p) := \int e^{ip\cdot x} \mu(dx) = e^{im\cdot x - \frac{1}{2} v x^2} \quad \forall_{p \in \mathbb{R}}$$

<u>Notation</u> $\mu = N(m,v)$

<u>Theorem 1</u> $(\Omega, \mathcal{O}\!\mathcal{l}, P)$ prob. space, $X_n : \Omega \to \mathbb{R}$ $(n \in \mathbb{N})$ indep. random variables with $X_n \sim N(0, v_n)$. Then the following statements are equivalent:

(i) $\sum_{n=1}^{\infty} X_n$ converges a.s.

(ii) $\sum X_n$ converges in probability.

(iii) $\sum X_n$ converges in L^2

(iv) $\sum v_n < \infty$

<u>Proof</u> (i), (iii) $\Rightarrow$ (ii) ✓

(ii) $\Rightarrow$ (iv). $\sum_{k=1}^{n} X_k \to S_\infty$ in prob.

$$\overset{\text{Dom. conv.}}{\Longrightarrow} E[e^{iS_\infty}] = \lim_{n\to\infty} \underbrace{E[e^{i\sum_{k=1}^{n} X_k}]}_{\overset{\text{indep.}}{=} \prod_{k=1}^{n} E[e^{iX_k}]} = e^{-\frac{1}{2}\sum_{k=1}^{\infty} v_k}$$

$\Rightarrow \sum v_k < \infty$

(iv) $\Rightarrow$ (iii): $\left\| \sum_{k=n}^{m} X_k \right\|_{L^2}^2 \overset{\text{indep.}}{=} \sum_{k=n}^{m} \|X_k\|_{L^2}^2 = \sum_{k=n}^{m} v_k$

$\longrightarrow 0$ as $n, m \to \infty$ if (iv) holds.

(iv) $\Rightarrow$ (i): Suppose (iv) holds, and let $S_\infty := L^2\text{-}\lim \sum_{k=1}^{n} X_k$.

$$\overset{X_k \text{ indep., centered}}{\Longrightarrow} E[S_\infty \mid X_1, \ldots, X_n] = \sum_{k=1}^{n} X_k$$

$\Longrightarrow$ partial sums form L^2 bounded martingale

$\Longrightarrow$ a.s. convergent

□

$$X \sim N(0,1) \overset{\text{if } \sigma \neq 0}{\Longleftrightarrow} m + \sigma X \sim N(m, \sigma^2)$$

Lemma 2 (Gaussian tail bound) $X \sim N(0,1) \Rightarrow$

$$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{t} - \frac{1}{t^3}\right)e^{-t^2/2} \le P[X \ge t] \le \frac{1}{\sqrt{2\pi}} \frac{1}{t} e^{-t^2/2} \quad \forall t > 0$$

Proof $\left(-\frac{1}{t}e^{-t^2/2}\right)' = \left(1 + \frac{1}{t^2}\right)e^{-t^2/2} \ge e^{-t^2/2}$

$$\left(\left(-\frac{1}{t} + \frac{1}{t^3}\right)e^{-t^2/2}\right)' = \left(1 - \frac{3}{t^4}\right)e^{-t^2/2} \le e^{-t^2/2}$$

Integrate $\Rightarrow$ Claim. $\square$

2.2. Probability measures on Banach spaces

$(E, \|\cdot\|)$ Separable Banach space,

i.e., there exists a countable dense subset.

$E^* :=$ all continuous linear functions $\ell : E \to \mathbb{R}$ "TOPOLOGICAL DUAL SPACE"

$$\|\ell\| = \sup\{\ell(x) : x \in E \text{ s.t. } \|x\| \le 1\}$$

__Fact__ 1) $(E^*, \|\cdot\|)$ is a Banach space

2) Riesz representation: If E is a Hilbert space then

$$\forall \ell \in E^* \ \exists! h \in E : \forall x \in E \ \ \ell(x) = (h, x)$$

$$\begin{aligned} i: E &\to E^* \\ h &\mapsto (h, \cdot) \end{aligned} \quad \text{is an isometry.}$$

__Lemma 3__ $\mathcal{B}(E) = \sigma(E^*)$

__Proof__: "$\supseteq$": $\ell \in E^* \Rightarrow \ell$ continuous $\Rightarrow \ell$ Borel-meas.

"$\subseteq$" Proof only for E^* separable:

$\{\ell_n : n \in \mathbb{N}\} \subseteq E^*$ dense

$$\Rightarrow \|x\| = \sup_{\substack{\ell \in E^* \\ \ell \neq 0}} \frac{\ell(x)}{\|\ell\|} = \sup_{n \in \mathbb{N}} \frac{\ell_n(x)}{\|\ell_n\|}$$

measurable w.r.t. $\sigma(E^*)$

$$\Rightarrow \quad B(x,r) \in \sigma(E^*) \quad \forall x \in E, r > 0$$

$$\Rightarrow \quad \mathcal{B}(E) \subseteq \sigma(E^*) \qquad \square$$

In particular $\mathcal{B}(E) = \sigma(Cyl)$ where

$$Cyl = \{ \{(\ell_1, \dots, \ell_n) \in B\} : n \in \mathbb{N}, \ell_1, \dots, \ell_n \in E^*, B \in \mathcal{B}(\mathbb{R}^n) \}$$

system of cylinder sets, $\cap$-stable

<u>Corollary 4</u> A prob. measure μ on $\mathcal{B}(E)$ is uniquely determined by its finite-dimensional marginals

$$\mu_{\ell_1, \dots, \ell_n} = \mu \circ (\ell_1, \dots, \ell_n)^{-1}$$

with $n \in \mathbb{N}$ and $\ell_1, \dots, \ell_n \in E^*$.

<u>Remark / Exercise</u> In Cor. 4, E^* can be replaced by any subset $K \subseteq E^*$ s.t. span(K) is dense in E^*

<u>Examples of Banach spaces and duals</u>

1) $E = L^p(S, \mathcal{S}, \nu)$ with $(S, \mathcal{S}, \nu)$ countably generated measure space

$p \in [1, \infty)$: E separable, $\frac{1}{p} + \frac{1}{q} = 1$:

$\forall \ell \in E^* \; \exists! \, g \in L^q(S, \mathcal{S}, \nu) : \ell(f) = \int f g \, d\nu \quad \forall f \in L^p(S, \mathcal{S}, \nu)$

Isometry $L^q(\nu) \to (L^p(\nu))^*$

$$g \mapsto \ell(f) = \int f g \, d\nu$$

2) $E = L^\infty(S, \mathcal{S}, \nu)$ not separable in general

3) $E = C_b(K)$ is separable if K is compact.

E^* not separable in general (signed measures)

However:

$$\mathcal{B}(E) = \sigma(\delta_s : s \in K), \quad \delta_s(f) = f(s)$$

$\Rightarrow$ A prob. measure on $\mathcal{B}(E)$ is uniquely determined by marginals

$$\mu \circ (\delta_{s_1}, \dots, \delta_{s_n})^{-1} = \text{Law of } x \mapsto (x(s_1), \dots, x(s_n))$$

4) $E = H^r(\mathbb{R}^d)$ = Sobolev space of order $(r, 2)$, $r \in \mathbb{R}$

$$H^r(\mathbb{R}^d)^* \cong H^{-r}(\mathbb{R}^d), \quad \text{cf. below.}$$

<u>Theorem 5</u> A prob. measure μ on $\mathcal{B}(E)$ is uniquely determined by its <u>characteristic function</u>

$$\hat{\mu}(\ell) = \int e^{i\ell(x)}\, \mu(dx) \quad , \quad \ell \in E^*.$$

<u>Proof</u>: $n \in \mathbb{N}$, $\ell_1, \dots, \ell_n \in E$

$\Rightarrow$ $\mu \circ (\ell_1, \dots, \ell_n)^{-1}$ has char. fct.

$$\varphi_{\ell_1, \dots, \ell_n}(p) = \int e^{i \sum_{i=1}^n p_i \ell_i(x)}\, \mu(dx) = \hat{\mu}\Big(\sum_{i=1}^n p_i \ell_i\Big), \; p \in \mathbb{R}^n$$

$\Rightarrow$ $\mu \circ (\ell_1, \dots, \ell_n)^{-1}$ uniquely determined by $\hat{\mu}$

Assertion follows by Cor. 4. $\square$

2.3. Gaussian measures on Banach spaces

$(E, \|\cdot\|)$ separable Banach space, μ prob. measure on $\mathcal{B}(E)$

Def. 1) μ is called a Gaussian measure

iff $\mu \circ \ell^{-1}$ is Gaussian for any $\ell \in E^*$.

2) The mean and covariance of μ are the linear resp. bilinear functions $m: E^* \to \mathbb{R}$ and $C: E^* \times E^* \to \mathbb{R}$ defined by

$$m(\ell) = \int \ell \, d\mu,$$

$$C(\ell, \tilde{\ell}) = \int (\ell - m(\ell))(\tilde{\ell} - m(\tilde{\ell}))\, d\mu$$

Rem. C is a non-neg. symm. bilinear form; uniquely determined by $C(\ell,\ell) = \mathrm{Var}_\mu(\ell)$

Theorem 6 μ is Gaussian with mean m and covariance C if and only if

$$\hat{\mu}(\ell) = e^{i m(\ell) - \frac{1}{2} C(\ell,\ell)} \quad \forall \ell \in E^*.$$

Notation: $\mu = N(m, C)$

Proof: μ Gaussian with m, C

$$\Longleftrightarrow \forall \ell \in E^* : \mu \circ \ell^{-1} = N(m(\ell), C(\ell,\ell))$$

$\square$

Def 1) A random variable $X : \Omega \to E$ defined on a probability space $(\Omega, \mathcal{O}\!\!\iota, P)$ is called <u>Gaussian</u> iff the law $P \circ X^{-1}$ is Gaussian (i.e. $\ell(X)$ is normally distributed for any $\ell \in E^*$)

2) A family $\{X_i : i \in I\}$ of random variables $X_i : \Omega \to E$ is called <u>jointly Gaussian</u> iff $(X_{i_1}, \dots, X_{i_n}) : \Omega \to E^n$ is Gaussian for any $n \in \mathbb{N}$ and $i_1, \dots, i_n \in I$

Lemma 7 Let $X_i : \Omega \to \mathbb{R}$ $(i \in I)$ jointly Gaussian. Then:

$$\mathrm{Cov}(X_i, X_j) = 0 \ \forall i \neq j \Rightarrow \{X_i : i \in I\} \text{ independent}$$

Proof W.l.o.g. $I = \{1, \dots, n\}$. If $X_1, \dots, X_n$ are jointly Gaussian then

$$E[e^{ip \cdot X}] = e^{ip \cdot m - \frac{1}{2} p \cdot Cp} \quad \forall p \in \mathbb{R}^n$$

with $m \in \mathbb{R}^n$ and $C \in \mathbb{R}^{n \times n}$ symmetric & non-negative.

Moreover, $C_{ij} = \mathrm{Cov}(X_i, X_j)$. Hence if the X_i are uncorrelated then

$$E[e^{ip \cdot X}] = \prod_{k=1}^{n} e^{ip_k m_k - \frac{1}{2} C_{kk} p_k^2} = \text{char. fct. of } \bigotimes_{k=1}^{n} N(m_k, C_{kk})$$

Now suppose that E is a <u>Hilbert space</u> with inner product $(\cdot,\cdot)$. Then we can identify $E^* \cong E$ by the Riesz isometry

$$x \in E \longmapsto x^* = (x,\cdot) \in E^*.$$

<u>Def.</u> A symmetric non-negative definite linear operator $A: E \to E$ is called <u>trace class</u> iff

$$\operatorname{tr}(A) := \sum_{n=1}^{\infty} (e_n, Ae_n) < \infty$$

w.r.t. an arbitrary complete ONB $\{e_n : n \in \mathbb{N}\}$ of E.

<u>Rem.</u> The value of $\operatorname{tr}(A)$ does not depend on the basis $\{e_n\}$.

<u>THEOREM 8</u> Suppose that μ is a Gaussian measure on a separable Hilbert space E. Then there exist a vector $a \in E$ and a trace class symmetric non-negative linear operator $K: E \to E$ such that

$$(*) \quad \int e^{i(p,x)}\, \mu(dx) = e^{i(p,a) - \frac{1}{2}(p,Kp)} \quad \forall p \in E.$$

Conversely, for any a, K as above, there is a unique Gaussian measure μ on H sat[illegible]

Rem. 1) a is called the <u>mean vector</u> of μ. The operator K is often called the "<u>covariance operator</u>" of μ. Note, however, that K depends in a substantial way on the space E where the measure has been realized and on the Riesz isometry on E.

2) $tr(K) = \int \|x-a\|^2 \mu(dx)$

<u>Proof</u> 1) μ Gaussian measure on E, $\varphi(p) := \int e^{i(p,x)} \mu(dx)$

$$\overset{\text{Thm } 6}{\underset{p^* := (p,\cdot)}{\Longrightarrow}} \quad \varphi(p) = \hat{\mu}(p^*) = e^{i m(p^*) - \frac{1}{2} C(p^*, p^*)}$$

$$= e^{i \bar{m}(p) - \frac{1}{2} \bar{C}(p,p)}, \quad \bar{m}(p) := m(p^*), \bar{C}(p,q) := C(p^*, q^*)$$

continuous in p by dominated convergence

$\Rightarrow$ $\bar{m}$ cont. linear fctl., $\bar{C}$ cont. symm. non-neg. def. bilinear

$\Rightarrow$ $\exists\, a \in E$, $K: E \to E$ bounded symmetric non-neg.:

$$\bar{m}(p) = (a, p), \quad \bar{C}(p,q) = (p, Kq)$$

<u>Claim</u>: $tr\, K = \int \|x-a\|^2 \mu(dx) < \infty$

To prove the claim we first show that K is a compact operator:

$$p_n \xrightarrow{w} 0 \Rightarrow (p_n, x) \to 0 \ \forall x \in E \overset{\text{dom. conv.}}{\Longrightarrow} \varphi(p_n) \to 1$$

$$\Rightarrow |\varphi(p_n)| = e^{-\frac{1}{2}(p_n, Kp_n)} \to 1$$

$$\Rightarrow \|K^{1/2} p_n\|^2 = (p_n, Kp_n) \to 0$$

Thus $K^{1/2}$ maps weakly conv. sequences to strongly conv sequences

$\overset{\text{Banach/Alaoglu}}{\Longrightarrow}$ $K^{1/2}$ maps bounded sequences to relatively compact sequences

$\Rightarrow$ $K^{1/2}$ compact linear operator, symmetric

$\overset{\text{Spectral Theorem}}{\Longrightarrow}$ $\exists$ complete ONB $\{e_n : n \in \mathbb{N}\}$ of E consisting of eigenvectors of $K^{1/2}$ (and hence of K)

By $(*)$, we then obtain:

$$\operatorname{tr} K = \sum_n (e_n, Ke_n) \overset{(*)}{=} \sum \operatorname{Var}(e_n^*)$$

$$= \sum \int (e_n, x-a)^2 \mu(dx)$$

$$\overset{\text{mon. conv.}}{=} \int \sum (e_n, x-a)^2 \mu(dx) \overset{\{e_n\}\text{ONB}}{=} \int \|x-a\|^2 \mu(dx)$$

The second moment on the right hand side is finite by <u>Fernique's Theorem</u> that will be proven in the next section.

2) Conversely let $a \in E$, $K: E \to E$ trace-class symm. non-neg. $\{e_n : n \in \mathbb{N}\}$ complete ONB, eigenvectors of K with λ_n

X_n ($n \in \mathbb{N}$) i.i.d. $\sim N(0,1)$

$$X(\omega) := a + \sum_{n=1}^{\infty} \sqrt{\lambda_n}\, X_n(\omega)\, e_n$$

Converges in $L^2(\Omega \to H, \mathcal{F}, P)$ since

$$\sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\, X_n e_n\|_{L^2}^2 = \sum \lambda_n\, E[X_n^2] = \operatorname{tr}(K) < \infty$$

X is Gaussian with mean vector a and covariance

$$C(p^*, p^*) = \operatorname{Var}((p, X)) = \operatorname{Var}\left(\sum \sqrt{\lambda_n}\,(p, e_n)\, X_n\right)$$

$$\overset{\text{indep.}}{=} \sum \lambda_n (p, e_n)^2 = (p, Kp)$$

$\square$

<u>Example (Finite dimensional case)</u> $E = \mathbb{R}^n$

$\forall a \in \mathbb{R}^n$ $\forall K \in \mathbb{R}^{n \times n}$ symmetric non-negative $\exists!$ Gaussian measure μ on E

s.t. $\int_{\mathbb{R}^n} e^{ip\cdot x}\, \mu(dx) = e^{ip\cdot a - \frac{1}{2} p\cdot Kp}$ $\forall p \in \mathbb{R}^n$, $\mu =: N(a, K)$

<u>Example</u> $E = \ell^2 = \{(x_n)_{n\in\mathbb{N}} : \sum x_n^2 < \infty\}$

(i) $\nexists$ Gaussian measure μ on E : $\int e^{i(p,x)}\mu(dx) = e^{-\frac{1}{2}\|p\|_{\ell^2}^2}$ $\forall p$

(ii) More generally let $v_n \in \mathbb{R}_+$, $n\in\mathbb{N}$. Is there a centered Gaussian measure μ on E with $Cov(x_n, x_m) = \delta_{nm}\cdot v_n$?

$$a = 0\ ,\quad K = \mathrm{Diag}(v_1, v_2, \dots)\ ,\ (Kp)_n = v_n p_n$$

$$\mu \text{ exists on } \ell^2 \overset{\text{Thm 2}}{\Longleftrightarrow} \operatorname{tr} K = \sum v_n < \infty$$

On the other hand, a measure as above always exists on $\mathbb{R}^\infty$: $\mu = \bigotimes_{n=1}^{\infty} N(0, v_n)$

$$\|x\|_{\ell^2}^2 = \sum x_n^2 < \infty \ \mu\text{-a.s.} \Leftrightarrow \sum v_n < \infty$$

<u>Remark</u> The second example essentially covers the general case, since by the Spectral Theorem any compact symmetric linear operator K on a separable Hilbert space can be represented as a diagonal matrix on ℓ^2 w.r.t. a complete orthonormal basis consisting of eigenvectors of K.

2.3.1. Fernique's Theorem $(E, \|\cdot\|)$ sep. Banach space

Theorem 8A (Fernique 1970)

μ Gaussian measure on $\mathcal{B}(E)$

$$\Rightarrow \exists \, \alpha > 0 : \int e^{\alpha \|x\|^2} \mu(dx) < \infty$$

Example 1) $(B_t)_{t \geq 0}$ BM$(\mathbb{R}^d)$, μ = Law of $(B_t)_{t \in [0,1]}$ is Gaussian measure on $C([0,1])$

$$\Rightarrow \exists \, \alpha > 0 : E\left[e^{\alpha \sup_{t \in [0,1]} |B_t|^2} < \infty \right]$$

follows also from reflection principle or Doob's inequality

2) $\mu = \bigotimes_{n=1}^{\infty} N(0, v_n)$ on $E = \ell^2$, $\sum v_n < \infty$

$$\int e^{\alpha \|x\|^2} \mu(dx) = \prod_{n=1}^{\infty} \underbrace{\int e^{\alpha x_n^2} N(0, v_n)(dx_n)}_{= \frac{1}{\sqrt{2\pi v_n}} \int e^{(\alpha - \frac{1}{2v_n}) u^2} du = \frac{1}{\sqrt{1 - 2\alpha v_n}}}$$

$$= \prod_{n=1}^{\infty} (1 - 2\alpha v_n)^{-1/2} \quad \text{provided } 2\alpha v_n < 1 \ \forall n$$

$$\sum v_n < \infty \Rightarrow \prod (1-2\alpha v_n) > 0 \quad \forall \alpha < \frac{1}{2 \sup v_n}$$

$$\Rightarrow \int e^{\alpha \|x\|^2} \mu(dx) < \infty \quad \text{"}$$

Proof of Fernique's Theorem:

<u>Lemma 8B</u> μ centered Gaussian, $R_\varphi : E \times E \to E \times E$ rotation

$$R_\varphi(x,y) := (x \sin\varphi + y \cos\varphi,\ x \cos\varphi - y \sin\varphi)$$

$$\Rightarrow (\mu \otimes \mu) \circ R_\varphi^{-1} = \mu \otimes \mu \quad \forall \varphi \in \mathbb{R}$$

<u>Proof</u> $\ell \in (E \times E)^* \Rightarrow \ell(x,y) = \ell(x,0) + \ell(0,y) = \ell_1(x) + \ell_2(y)$, $\ell_1, \ell_2 \in E^*$

$$\Rightarrow \widehat{\mu \otimes \mu}(\ell) = \iint e^{i\ell(x,y)} \mu(dx)\,\mu(dy) = \hat\mu(\ell_1)\,\hat\mu(\ell_2)$$

$$= e^{-\frac{1}{2}(C(\ell_1,\ell_1) + C(\ell_2,\ell_2))}$$

On the other hand:

$$\widehat{(\mu \otimes \mu) \circ R_\varphi^{-1}}(\ell) = \iint e^{i\overbrace{\ell(R_\varphi(x,y))}^{=\ell_1(x\sin\varphi + y\cos\varphi) + \ell_2(x\cos\varphi - y\sin\varphi)}} \mu(dx)\,\mu(dy) \overset{\text{Lin.}}{=} \hat\mu(\tilde\ell_1)\,\hat\mu(\tilde\ell_2)$$

$$= e^{-\frac{1}{2}(C(\tilde\ell_1,\tilde\ell_1) + C(\tilde\ell_2,\tilde\ell_2))}$$ where $\tilde\ell_1 := \sin\varphi\,\ell_1 + \cos\varphi\,\ell_2$, $\tilde\ell_2 := \cos\varphi\,\ell_1 - \sin\varphi\,\ell_2$

Bilinearity & symmetry $\Rightarrow C(\tilde{\ell}_1,\tilde{\ell}_1)+C(\tilde{\ell}_2,\tilde{\ell}_2) = \overbrace{\left[(\sin\varphi)^2+(\cos\varphi)^2\right]}^{=1}\left[C(\ell_1,\ell_1)+C(\ell_2,\ell_2)\right]$

$$\Rightarrow (\mu\otimes\mu)\circ R_\varphi^{-1} = \mu\otimes\mu \qquad \square$$

<u>Strengthened form of Fernique's Theorem:</u>

<u>Theorem 8C</u> There exist <u>universal</u> constants $c>0$ and $C\in(0,\infty)$ s.t.

$$(*) \int \exp\left(\frac{c}{\tau^2}\|x\|^2\right)\mu(dx) \le C$$

for any Gaussian measure μ on a separable Banach space E, and for any $\tau\in\mathbb{R}_+$ such that $\mu(\|x\|>\tau)\le 1/4$.

<u>Rem.</u> Actually, $(*)$ holds for any prob. measure μ on $\mathcal{B}(E)$ satisfying $(\mu\otimes\mu)\circ R_{\pi/4}^{-1} = \mu\otimes\mu$, cf. the proof below.

Bilinearity & symmetry $\Rightarrow C(\widehat{\ell_1},\widehat{\ell_1})+C(\widehat{\ell_2},\widehat{\ell_2}) = \underbrace{\left((\sin\varphi)^2+(\cos\varphi)^2\right)}_{=1}\left(C(\ell_1,\ell_1)+C(\ell_2,\ell_2)\right)$

$$\Rightarrow \widehat{(\mu\otimes\mu)\circ R_\varphi^{-1}} = \widehat{\mu\otimes\mu} \; . \quad \square$$

<u>Proof of Theorem 8C for centered Gaussian measures</u>: GOAL: $\mu(\|x\|>t) \leq e^{-2\alpha t^2}$ for large t.

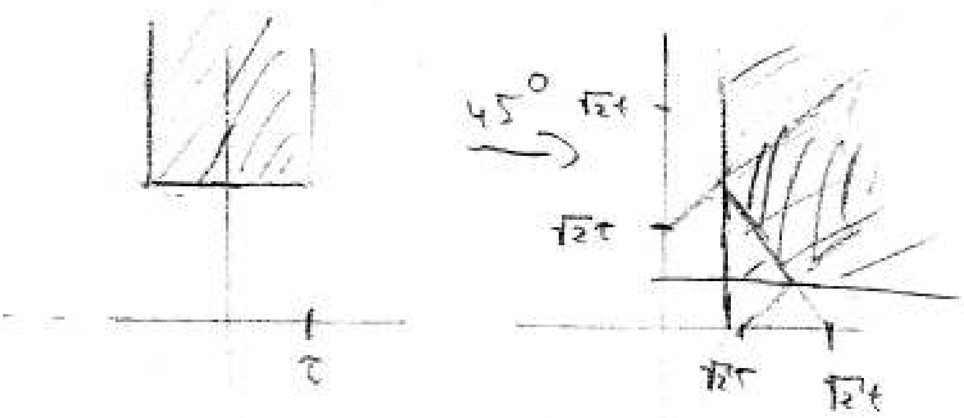
1) Let $t>\tau>0$. Then:

$$\mu(\|x\|\leq\tau)\,\mu(\|x\|>t) = \mu\otimes\mu\left(\{(x,y): \|x\|\leq\tau, \|y\|>t\}\right)$$

$$\overset{\text{Lem. 8B}}{\underset{\text{Rot. }45^0}{=}} \mu\otimes\mu\left(\left\{(x,y): \left\|\tfrac{x-y}{\sqrt{2}}\right\|\leq\tau, \left\|\tfrac{x+y}{\sqrt{2}}\right\|>t\right\}\right)$$

$$\leq \mu\otimes\mu\left(\min(\|x\|,\|y\|)>\tfrac{t-\tau}{\sqrt{2}}\right) = \mu\left(\|x\|>\tfrac{t-\tau}{\sqrt{2}}\right)^2$$

since $\min(\|x\|,\|y\|) \geq \frac{1}{2}\big(\underbrace{\|x+y\|}_{\geq\sqrt{2}t} - \underbrace{\|x-y\|}_{\leq\sqrt{2}\tau}\big) \geq \frac{t-\tau}{\sqrt{2}}$ on set.



$$\Rightarrow \mu(\|x\|>t) \le \frac{4}{3}\,\mu\Big(\|x\| > \frac{t-\tau}{\sqrt{2}}\Big)^2 \quad \forall t>\tau$$

where τ is chosen s.t. $\mu(\|x\|\le\tau) \ge 3/4$.

Now iterate the estimate: $t_0=\tau$, $t_{n+1}=\tau+\sqrt{2}\,t_n$

$$\Rightarrow \mu(\|x\|>t_n) \le \frac{1}{4}\,3^{1-2^n} \quad \forall n\in\mathbb{N}$$

$$\overset{t_{n+1}\le 2^{n/2}5\tau}{\Rightarrow} \exists\, c>0:\ \mu(\|x\|>t) \le e^{-2ct^2/\tau^2} \quad \forall t\ge\tau\ \forall\tau$$

$$\Rightarrow \int e^{c\|x\|^2/\tau^2}\,\mu(dx) \le e^c + \int_\tau^\infty \frac{2ct}{\tau^2}\,e^{ct^2/\tau^2}\,\mu(\|x\|>t)\,dt$$

$$= e^c + \int_1^\infty t\,e^{-ct^2}\,dt =: C < \infty \quad \square$$

Extension to non-centered Gaussian measures

Fact: E separable, μ Gaussian on $\mathcal{B}(E)$ $\Rightarrow \exists\, a\in E:\ m(\ell)=\ell(a)\ \forall \ell\in$

a = mean vector of μ

Proof for Hilbert spaces cf. above, general case cf. e.g. [Bogachev]

2.13f

$\tau_a(x) = a + x$ translation by a

$\Rightarrow \mu \circ \tau_{-a}^{-1} =: \mu_0$ is centered Gaussian

$$\Rightarrow \int e^{\alpha \|x\|^2} \mu(dx) = \int e^{\alpha \|a+x\|^2} \mu_0(dx)$$

$$\leq e^{2\alpha \|a\|^2} \int e^{2\alpha \|x\|^2} \mu_0(dx) < \infty$$

Corollary 8D (Moment bounds) $Z: \Omega \to E$ Gaussian r.v. Then:

1) $$E\left[\exp\left(\frac{c}{16} \frac{\|Z\|^2}{E[\|Z\|]^2}\right)\right] \leq C$$

2) $$E[\|Z\|^{2n}] \leq C\, n! \left(\frac{16}{c}\right)^n E[\|Z\|]^{2n} \quad \forall n \in \mathbb{N}$$

Proof 1) Markov: $P[\|Z\| > \tau] \leq \frac{1}{\tau} E[\|Z\|] \leq \frac{1}{4}$

if $\tau := 4\, E[\|Z\|]$. Now apply Thm. 8C.

2) $e^{\frac{c}{16} x^2} \geq \left(\frac{c}{16}\right)^n x^{2n} / n!$ Hence 1) $\Rightarrow$ 2). $\square$

2.4. Standard normal distributions

$(H, (\cdot,\cdot))$ separable Hilbert space,

$(E, \|\cdot\|)$ Banach space s.t. $H \subseteq E$ densely & continuously embedded

Then: $$E^* \subseteq H^* \cong H \subseteq E$$

$\uparrow$ Riesz isometry $h \mapsto h^* = (h, \cdot)$

GOAL: "Standard normal distribution" ν w.r.t. $(\cdot,\cdot)$

$Cyl(E) :=$ all cylinder sets $\{x \in E : (\ell_1(x), \dots, \ell_n(x)) \in B\}$ with $n \in \mathbb{N}$, $B \in \mathcal{B}(\mathbb{R}^n)$, $\ell_1, \dots, \ell_n \in E^*$

$Cyl(H) = \{P^{-1}(B) : n \in \mathbb{N},\ B \in \mathcal{B}(\mathbb{R}^n),\ P: H \to \mathbb{R}^n \text{ orth. proj.}\}$

For cylinder sets on H we can define:

$$(*) \quad \nu(P^{-1}(B)) := N(0, I_n)(B)$$

whenever $P: H \to \mathbb{R}^n$ is an orthogonal projection.

<u>Exercise</u> ν is well defined (independent of choice of P)

Lemma 9 1) ν is a non-negative additive function (a "cylindrical measure") on the algebra $Cyl(H)$.

2) If $\dim H = \infty$ then ν does not extend to a σ-additive measure on $\mathcal{B}(H)$.

Proof 1) by σ-additivity of $N(0, I_n)$, Exercise

2) Consequence of Theorem 8 ($\nexists$ Gaussian measure with $K = I$) □

Alternative proof of non-existance of a standard normal distribution on H: If ν had a σ-additive extension $\hat{\nu}$ on $\mathcal{B}(H)$ then this would be unique, and hence

$$\hat{\nu}(U^{-1}(B)) = \hat{\nu}(B) \quad \forall B \in \mathcal{B}(H), U: H \to H \text{ isometry}$$

This contradicts the following fact:

Theorem 10 If $\dim H = \infty$ then δ_0 is the only probability measure on $\mathcal{B}(H)$ that is invariant under rotations;

<u>Proof</u> μ prob. measure on $\mathcal{B}(H)$, $\mu \neq \delta_0$

$$\Rightarrow 0 < \mu(\underbrace{H \setminus \{0\}}_{= \bigcup B(x_n, \|x_n\|_H/2)}) \leq \sum_n \mu(B(x_n, \|x\|/2))$$

for countable subset $\{x_n\} \subseteq H \setminus \{0\}$

$$\Rightarrow \exists\, x \in H \setminus \{0\} : \underbrace{\mu(B(x, \|x\|/2))}_{=:\varepsilon} > 0$$

Choose complete ONB $\{e_n\}$ s.t. $x = r e_1$, $r = \|x\|$.

$$\overset{\text{rotation invariance}}{\Longrightarrow} \mu(\underbrace{B(re_n, r/2)}_{\text{disjoint}}) = \mu(B(re_1, r/2)) = \varepsilon \quad \forall n$$

$$\Rightarrow \mu(H) \geq \sum \mu(B(re_n, r/2)) = \infty \quad ↯ \qquad \square$$

<u>Way out</u>: $H \subsetneqq E \Rightarrow E^* \subsetneqq H^* \Rightarrow \mathcal{C}yl(E) \subsetneqq \mathcal{C}yl(H)$

Therefore, $(\nu, \mathcal{C}yl(E))$ may have a σ-additive extension to $\mathcal{B}(E)$ although $(\nu, \mathcal{C}yl(H))$ does not have a σ-additive extension!

$$E^* \subseteq H^* \cong H \subseteq E \quad \text{densely and continuously}$$

<u>Def.</u> 1) A centered Gaussian measure ν on $\mathcal{B}(E)$ is called a <u>$(\cdot,\cdot)_H$ standard normal distribution</u> iff

$$(*) \quad C(\ell,\ell) = \mathrm{Var}_\nu(\ell) = (\ell,\ell)_{H^*} = (h,h)_H$$

holds for any $\ell \in E^*$ and $h \in H$ s.t. $\ell(x) = (h,x)_H \ \forall x \in H$.

2) The triple (E,H,ν) is then called an <u>abstract Wiener space</u>.

<u>Examples</u> 1) $E = H = \mathbb{R}^n$, $\nu = N(0, I_n)$

2) $E = \{x \in C([0,1]) : x(0) = 0\}$ with sup norm,

$H = \{x \in E : x \text{ abs. cont. with } x' \in L^2\}$, $(x,y)_H = \int_0^1 x'(s) y'(s)\, ds$,

$\nu =$ Wiener measure on E

(E,H,ν) is an abstract Wiener space, since

$$\mathrm{Cov}_\nu(\delta_s, \delta_t) = \min(s,t) = \int_0^1 I_{(0,s)} I_{(0,t)} = (h_s, h_t)_H \quad \forall s,t \in [0,1]$$

where $h_s(r) = \int_0^r I_{(0,s)} = r \wedge s$ is the elem. in H corresponding to δ_s:

$$(h_s, x)_H = \int_0^s x'(r)\, dr = x(s) = \delta_s(x) \quad \forall x \in H.$$

Construction of the measure ν:

$\{e_n : n \in \mathbb{N}\}$ complete ONB of H

$Z_n : \Omega \to \mathbb{R}$ $(n \in \mathbb{N})$ i.i.d. $\sim N(0,1)$ on $(\Omega, \mathcal{A}, P)$

Theorem 11 If the series

$$S(\omega) = \sum_{n=1}^{\infty} Z_n(\omega)\, e_n$$

converges weakly in E for P-a.e. $\omega \in \Omega$, then (E, H, ν) with $\nu := P \circ S^{-1}$ is an abstract Wiener space.

Proof Let $\ell \in E^*$. Then

$$\ell(S) = \sum_{n=1}^{\infty} Z_n\, \ell(e_n)$$

is centered Gaussian with variance

$$C(\ell,\ell) = \mathrm{Var}_\nu(\ell) = \mathrm{Var}(\ell(S)) = \sum_{n=1}^{\infty} \ell(e_n)^2$$

$$= \sum (h, e_n)^2 = \|h\|_H^2 \quad \text{if } \ell(x) = (h,x)_H \ \forall x \in H.$$

$\Box$

Example (Wiener measure) E, H as in Example 2 above,

$e_0(t) = t$, $e_{n,k}(t)$ Schauder functions form ONB of H

$Z_0, Z_{n,k}$ i.i.d. $\sim N(0,1) \Rightarrow S = Z_0 e_0(t) + \sum_{n,k} Z_{n,k} e_{n,k}(t)$ a.s. unif. conv. on $[0,$

General criterion for existence of ν on E:

~~Assume first that E~~ is a separable Hilbert space.

Def. A bounded linear operator $B: H \to E$ is called Hilbert-Schmidt iff $\sum_{n=1}^{\infty} \|Be_n\|_E^2 < \infty$ for some (or, equivalently, for any) ONB $\{e_n : n \in \mathbb{N}\}$ of H.

Rem. $B: H \to E$ bounded $\Rightarrow$ $B^*: E(\cong E^*) \to H(\cong H^*)$ bounded

$$(Bx, y)_E = (x, B^*y)_H \quad \forall x \in H, y \in E$$

$$\sum_{n=1}^{\infty} \|Be_n\|_E^2 = \sum (e_n, B^*Be_n)_H = \operatorname{tr}(B^*B)$$

Hence B is Hilbert-Schmidt if and only if B^*B is trace class. In particular independent of choice of ONB!

Theorem 12 Suppose that E is a Hilbert space s.t. $H \subseteq E$ densely, continuously.

Then there exists an $(\cdot,\cdot)_H$ standard normal distribution ν on E if and only if the ~~embedding~~ inclusion map $i: H \hookrightarrow E$ is Hilbert-Schmidt (i.e. $\sum \|e_n\|_E^2 < \infty$)

Remark 1) $(x,y)_E = (ix, iy)_E = (x, Ay)_H \; \forall x,y \in H$ where $A = i^* i$.

In particular i is Hilbert-Schmidt iff A is trace-class.

2) A is the covariance operator of ν on E, cf. the proof below.

consisting of eigenvectors of A

Proof "$\Leftarrow$" $\{e_n\}$ ONB of H, Z_n i.i.d. $\sim N(0,1)$. Then:

$$S = \sum_{n=1}^{\infty} Z_n e_n \text{ converges a.s. in } E$$

$(e_n, e_m)_E = (e_n, Ae_m)_H = 0$ for $n \neq m$

$$\Longleftrightarrow \sum_{n=1}^{\infty} \|e_n\|_E^2 < \infty \Longleftrightarrow i \text{ Hilbert-Schmidt}$$

In this case, $(E, H, P \circ S^{-1})$ is abstract Wiener space.

"$\Rightarrow$" Suppose (E, H, ν) is abstract Wiener space.

Computation of "covariance operator" on E:

$$p \in \mathbb{H} \Rightarrow (p, x)_E = (Ap, x)_H \quad \forall x \in H$$

$$\mathrm{Cov}\big((p,\cdot)_E, (q,\cdot)_E\big) \overset{(*)}{=} (Ap, Aq)_H = (p, Aq) \quad \forall p, q \in \mathbb{H}$$

$H \subseteq$ dense

$$\Rightarrow A \text{ is covariance operator of } \nu \text{ on } E$$

$$\overset{\text{Thm 8}}{\Rightarrow} \mathrm{tr}(A) < \infty \overset{\text{Rem.}}{\Rightarrow} i \text{ Hilbert-Schmidt.}$$

$\square$

Covariance operator on H: I

Covariance operator on E: $A = i^* i$ trace-class

Examples 1) $H = \ell^2 = \{(x_n) : \sum x_n^2 < \infty\}$, $(x,y)_H = \sum x_n y_n$,

$$E = \{(x_n) : \sum w_n x_n^2 < \infty\}, \quad (x,y)_E = \sum w_n x_n y_n .$$

Then ν_H exists on $E \iff \infty > \sum \|e_n\|_E^2 = \sum w_n$

2) Wiener measure revisited : $H = \{x : [0,1] \to \mathbb{R} : x(0) = 0, x \text{ abs. cont. with } x' \in L^2\}$

$$(x,y)_H = \int_0^1 x'(s)\, y'(s)\, ds$$

ν_H exists on $E = \{x \in C([0,1]) : x(0) = 0\}$ and on $\hat{E} = L^2(0,1)$.

However :

$\nexists$ Hilbert space $\tilde{E}$ s.t. $H \subseteq \tilde{E} \subseteq E$ densely & continuously and $(\tilde{E}, H, \nu)$ is an abstract Wiener space for some measure ν.

Sketch of proof (cf. Bogachev 3.6.7):

$\tilde{E} \subseteq E$ continuously $\Rightarrow$ embedding $j : \tilde{E} \to C([0,1])$ bounded lin. operator

$\overset{(\dots)}{\Longrightarrow}$ embedding $\hat{j} : \tilde{E} \to L^2(0,1)$ Hilbert-Schmidt

Hence:

ν_H $(\cdot,\cdot)_H$ standard normal on $\tilde{E}$ $\Rightarrow$ $i : H \to \tilde{E}$ Hilbert-Schmidt

$\Rightarrow \underset{HS}{\hat{j}} \circ \underset{HS}{i} : H \to L^2(0,1)$ trace class $\lightning$ $\sum \|e_n\|_{L^2}^2 = \sum \frac{1}{n} = \infty$

Generalization to Banach spaces

E separable Banach space

$i: H \hookrightarrow E$ contin. embedding with dense range

$\nu: \mathcal{Z}\ell(H) \to [0,1]$ cylindrical standard normal distribution on H

Theorem 13 (L. Gross 1965)

$\nu|_{\mathcal{Z}\ell(E)}$ extends to a probability measure on $\mathcal{B}(E)$

if and only if for every $\varepsilon > 0$ there exists an orthogonal projection $P: H \to H$ with finite dimensional range s.t.

$$\nu(\|\tilde{P}x\| > \varepsilon) < \varepsilon \quad \forall \text{ fin. dim. orth. proj. } \tilde{P}: H \to H \text{ s.t. } \mathrm{Ran}(\tilde{P}) \perp \mathrm{Ran}(P)$$

In this case, (E, H, ν) is an abstract Wiener space.

Proof: see Bogachev: Gaussian measures, 3.9.5.

2.5. Gaussian measures on Sobolev spaces

Ref. Sheffield: Gaussian free fields for mathematicians, arXiv

$D \subseteq \mathbb{R}^d$ open

Δ_D self-adjoint realization of Laplacian with Dirichlet boundary conditions on $L^2(D)$

Spectral resolution: $\mathrm{Spec}(-\Delta_D) \subseteq (0,\infty)$

$$-\Delta_D x = \sum \lambda_n (e_n, x) e_n, \quad \mathrm{Dom}(\Delta_D) = \{x \in L^2(D) : \sum \lambda_n^2 (e_n, x)^2 < \infty\}$$

if D bounded, $\{e_n\}$ ONB of eigenvectors, $\lambda_n > 0$ eigenvalues

$$-\Delta_D = \int \lambda \, P(d\lambda), \quad \mathrm{Dom}(\Delta_D) = \{x : \int \lambda^2 (x, P(d\lambda) x) < \infty\}$$

in general, where P is projection valued measure

Sobolev inner products. Let $r \in \mathbb{R}$, $\alpha > 0$.

$$(x,y)_r := \left((\alpha - \Delta_D)^{r/2} x, (\alpha - \Delta_D)^{r/2} y \right)_{L^2(D)}$$

$$= \left(x, (\alpha - \Delta_D)^r y \right)_{L^2(D)} \quad \text{if } y \in \mathrm{Dom}\left((\alpha-\Delta_D)^r\right),$$

$$= \sum (\alpha + \lambda_n)^r (x, e_n)_{L^2} (y, e_n)_{L^2} \quad \text{if } \mathrm{Spec}(-\Delta_D) \text{ discrete},$$

$r \geq 0$: $(\cdot,\cdot)_r$ inner product on Hilbert space

$$H_r := \mathrm{Dom}\left((\alpha-\Delta_D)^{r/2}\right) = \{x \in L^2(D) : \sum (\alpha+\lambda_n)^r (x, e_n)^2 < \infty\}$$

<u>Rem.</u> 1) $(x,y)_1 \overset{\text{ibp.}}{=} \int_D (\alpha\, xy + \nabla x \cdot \nabla y)$

2) More generally: D bounded or $\alpha > 0$, $r \in \mathbb{N}$

$$\Rightarrow \quad \|x\|_r^2 \approx \sum_{|\beta| \le r} \|\partial^\beta x\|^2_{L^2(D)} = \|x\|^2_{H^{r,2}(D)}$$

H_r is Sobolev space of order $(r,2)$

$H_0 = L^2(D)$, $H_1 = H_0^{1,2}(D)$, $H_2 = \mathrm{Dom}(-\Delta_D) \subseteq H^{2,2}(D)$

<u>Def.</u> A $(\cdot,\cdot)_r$ standard normal distribution is called

1) <u>White noise</u> for $r = 0$
2) <u>Colored noise</u> for $r > 0$
3) <u>Gaussian free field</u> for $r = 1$ ($\alpha = m^2$, m mass)

Realization of ν_r on Hilbert space?

<u>Negative-order Sobolev spaces</u> $r \ge 0$, D bounded or $\alpha > 0$

$(\alpha - \Delta_D)^{-r/2}$ bounded linear op., spectrum $\subseteq [0, (\inf \operatorname{spec}(\alpha-\Delta_D))^{-r/2}]$

$\|x\|_{-r} = \|(\alpha - \Delta_D)^{-r/2} x\|_{L^2(D)}$ weaker than L^2 norm

$H_{-r} :=$ completion of $L^2(D)$ w.r.t. $\|\cdot\|_{-r}$

$H_r \subseteq L^2(D) \subseteq H_{-r}$ densely & continuously

<u>Lemma 14</u> $\|x\|_{-r} = \sup\{(x,y)_{L^2} : y \in H_r \text{ with } \|y\|_r \leq 1\}$ $\forall x \in L^2(D)$

dual norm of $\|\cdot\|_r$ w.r.t. L^2 inner product.

<u>Proof</u> $(x,y)_{L^2} = \left((\alpha-\Delta)^{-r/2}x, (\alpha-\Delta)^{r/2}y\right)_{L^2}$

$$\overset{CS}{\leq} \|x\|_{-r}\|y\|_r \quad \forall x \in L^2, y \in H_r$$

Moreover, equality holds for $(\alpha-\Delta)^{r/2}y = \text{const.}\,(\alpha-\Delta)^{-r/2}x$, i.e., for $y = \text{const.}\,(\alpha-\Delta)^{-r}x$. $\square$

<u>Consequence</u>: $H_r^* \cong H_{-r}$ in the sense that

$\forall \ell \in H_r^* \ \exists!\ \xi \in H_{-r} : \ell(y) = \langle \xi, y\rangle \quad \forall y \in H_r$

where $\xi \mapsto \langle \xi, y\rangle$ is the unique continuous extension of the L^2 inner product from $L^2(D)$ to H_{-r}.

$$L^2(D) \ni \xi \longmapsto (\xi,\cdot)_{L^2} \in H_r^*$$

extends to isometry

$$H_{-r} \ni \xi \longmapsto \langle \xi, \cdot\rangle \in H_r^*$$

H_{-r} consists of <u>distributions</u>, i.e., of continuous linear functionals $\langle \xi, \cdot \rangle : H_r \supseteq C_0^\infty(D) \to \mathbb{R}$.

<u>Example</u> $D = (0,1)^d$. Eigenfunctions of $-\Delta_D$:

$$e_n(s) = \prod_{k=1}^{d} \sin(n_k \pi s_k) / Z_n \quad , \quad n = (n_1, \dots, n_d) \in \mathbb{N}^d$$

$$\lambda_n = (n_1^2 + \dots + n_d^2)\pi^2 = |n|^2 \pi^2$$

$$\|x\|_r^2 = \sum_{n_1} \cdots \sum_{n_d} (\alpha + |n|^2 \pi^2)^r \underbrace{(x, e_n)_{L^2}^2}_{\text{Fourier coefficients of } x}$$

$\#$ eigenvalues $\leq \lambda$: $|\{n \in \mathbb{N}^d : |n|^2 \leq \lambda/\pi^2\}| \sim$

$$\overset{\lambda \uparrow \infty}{\sim} \frac{1}{2^d} \mathrm{Vol}\left(B(0, \sqrt{\lambda}/\pi) \right) = c_d \cdot \lambda^{d/2}$$

True for any bounded domain!

<u>FACT. Weyl's formula</u> $D \subset \mathbb{R}^d$ bounded, $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$ eigenvalues of $-\Delta_D$ $\Rightarrow$ $\lambda_k \sim \text{const.} \cdot k^{2/d}$ as $\lambda \to \infty$

Hilbert-Schmidt embeddings

$r \geq s \Rightarrow i: H_r \subseteq H_s$ dense & continuous embedding

Realization of ν_r on H_s? ok iff i is Hilbert-Schmidt!

THEOREM 15 Suppose that D is bounded. Then:

$$i: H_r \subseteq H_s \text{ Hilbert-Schmidt} \Leftrightarrow s < r - \frac{d}{2}$$

Proof: $\|x\|_s^2 = \|(\alpha-\Delta)^{s/2}x\|_{L^2} = \|(\alpha-\Delta)^{r/2}(\alpha-\Delta)^{(s-r)/2}x\|_{L^2}$

$$= \|(\alpha-\Delta)^{(s-r)/2}x\|_r \qquad \forall x \in H_r$$

$\{e_k\}$ ONB of H_r s.t. $-\Delta e_k = \lambda_k e_k \Rightarrow$

$$\sum \|e_k\|_s^2 = \sum \underbrace{(\alpha+\lambda_k)}_{\sim \text{const.}\, k^{2/d} \text{ by Weyl's lemma}}{}^{s-r} \sim \sum k^{2(s-r)/d}$$

$$< \infty \Leftrightarrow s-r < -d/2 \qquad \square$$

Example 1) $d=1$: $H_1 \subseteq H_s$ Hilbert-Schmidt $\Leftrightarrow s < 1/2$

$\rightarrow$ Gaussian free field exists on H^s

2) $d=2$: $H_1 \subseteq H_s$ Hilbert-Schmidt $\Leftrightarrow s < 0$

$\rightarrow$ Gaussian free field does not exist on $H^0 = L^2$, exists on $H^{-\varepsilon}$ ver

3) $L^2 \subset H_s$ Hilbert-Schmidt $\iff s < -\frac{d}{2}$

$\rightsquigarrow$ white noise takes in values in $H^{-\frac{d}{2}-\varepsilon}$

4) $H_r \subset L^2$ Hilbert-Schmidt $\iff r > \frac{d}{2}$

$\rightsquigarrow$ For any $r > \frac{d}{2}$ the $(\cdot,\cdot)_r$ standard normal distribution exists on $L^2(D)$
(i.e. a typical sample is a square-integrable function and not a distribution)

5) $r > d+m \implies \exists\, s > \frac{d}{2}+m : r > s + \frac{d}{2}$

SOBOLEV EMBEDDING $\implies$ $H_r \subset H_s$ Hilbert-Schmidt, $H_s \subset C^{m,\alpha}(\overline{D})$ compact $\forall \alpha < s - \frac{d}{2} -$

$\rightsquigarrow$ For any $r > d+m$ the $(\cdot,\cdot)_r$ standard normal distribution exists on $C^{m,\alpha}(\overline{D})$

<u>Exercise</u> Prove that for $d = 1$, the Gaussian free field on $[0,1]$ is a Brownian bridge.

2.6. Gaussian Random Fields

D : index set, T sep. Banach space

e.g. $D \subseteq \mathbb{R}^d$, $T = \mathbb{R}$.

Def. 1) A <u>random field</u> is a collection $(X_s)_{s\in D}$ of random variables $X_s : \Omega \to T$ defined on a joint prob. space $(\Omega, \mathfrak{A}, P)$

2) A random field $(X_s)_{s\in D}$ is called <u>Gaussian</u> iff $P\circ(X_{s_1}, \dots, X_{s_n})^{-1}$ is Gaussian for any $n \in \mathbb{N}$ and $s_1, \dots, s_n \in D$.

Gaussian random field = Gaussian process indexed by D.

Law $P\circ X^{-1}$ of GRF is Gaussian measure on product space
$$T^D = \{x : D \to T\} \text{ with product } \sigma\text{-algebra } \mathcal{F} = \sigma(x \mapsto x_s : s\in D)$$

Now assume $T = \mathbb{R}^n$. Then, $P\circ X^{-1}$ is uniquely determined by
$$m(s) = E[X_s] \in \mathbb{R}^n, \quad s \in D,$$
$$C(s,t) = \mathrm{Cov}(X_s, X_t) \in \mathbb{R}^{n\times n}, \quad s,t \in D.$$

THEOREM 16 Let $m : D \to \mathbb{R}^n$ and $C : D\times D \to \mathbb{R}^{n\times n}$ with $C^{ij}(s,t) = C^{ji}(s,t)$ and $C^{ij}(s,t) = C^{ij}(t,s)$ $\forall i,j \in \{1,\dots,n\}, s,t\in D$, be given. Then there exists a GRF with mean m and covariance C if and only if the function C is <u>non-neg. definite</u>, i.e., $\sum_{k,l=1}^{m} \xi_k \cdot C(s_k, s_l)\xi_l \geq 0$ $\forall m \in \mathbb{N}, s_1,\dots,s_m \in D, \xi_1,\dots,\xi_m \in \mathbb{R}^n$

Proof C non-negative def

$\iff$ $\forall s_1, \dots, s_\ell$ $\exists$ Gaussian measure on $\mathbb{R}^{n \cdot n}$ with mean $(m(s_i))_{i=1}^{\ell}$ and covariance matrix $(C(s_i, s_j))_{i,j=1}^{\ell}$

consistent $\iff$ Kolmogorov extension thm. $\exists$ GRF (m, C) on $\Omega = (\mathbb{R}^n)^D$ $\square$

EXAMPLE: **WHITE NOISE** $(S, \mathcal{S}, \nu)$ finite measure space, $n=1$

Def. A centered Gaussian field $(W(B))_{B \in \mathcal{S}}$ is called a white noise ("Gaussian random measure") on $(S, \mathcal{S}, \nu)$ iff

$$\mathrm{Cov}(W(A), W(B)) = \nu(A \cap B) \quad \forall A, B \in \mathcal{S}$$

Properties: (i) A_i $(i \in I)$ disj. $\Rightarrow$ $W(A_i)$ independent r.v.

(ii) A_i $(i \in \mathbb{N})$ disj. $\Rightarrow$ $W(\bigcup A_i) = \sum W(A_i)$ a.s.

(iii) $W(\emptyset) = 0$

Important remark: In general, $A \mapsto W(A)(\omega)$ is not a signed measure, because a) exceptional set in (ii) may depend on A_i; b) total variation may be a.s. infinite

Definition extends to σ-finite ν by restricting to sets B with $\nu(B) < \infty$

Ex. $(B_t)_{t\in\mathbb{R}_+}$ standard Brownian motion

$\Rightarrow \exists!$ (up to modifications) white noise on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), \lambda)$
s.t.

$$W((s,t]) = B_t - B_s \quad \forall\, 0 \le s < t$$

"distribution function of W, $W = \dot{B}$"
a.s. not of finite variation!

Extension from sets to functions:

$$f = \sum_{i=1}^n c_i I_{A_i}, \quad n\in\mathbb{N},\ c_i \in \mathbb{R},\ A_i \in \mathcal{S}$$

$$W(f) := \sum_{i=1}^n c_i W(A_i) \quad \text{almost surely well-defined by (ii),}$$

centered Gaussian r.v. with

$$E[W(f)^2] = Var(W(f)) \overset{A_i \text{ disjoint (w.l.o.g.)}}{=} \sum_{i=1}^n c_i^2 Var(W(A_i))$$
$$= \sum_{i=1}^n c_i^2 \nu(A_i) = \int f^2 d\nu$$

i.e.

$$f \longmapsto W(f)$$

$$\text{Elem. Functions} \subseteq L^2(S,\mathcal{S},\nu) \longrightarrow L^2(\Omega,\mathcal{A},P) \qquad \text{ISOMETRY}$$

$\Rightarrow \exists!$ extension to isometry $L^2(S,\mathcal{S},\nu) \to L^2(\Omega,\mathcal{A},P)$

$\left(W(f)\right)_{f\in L^2(S,\mathcal{S},\nu)}$ is centered GRF with

$$\mathrm{Cov}\left(W(f),W(g)\right) = (f,g)_{L^2(\nu)} \quad \forall f,g\in L^2(\nu)$$

Example: $(S,\mathcal{S},\nu)=(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+),\lambda)$, $W((0,t])=B_t$ standard BM

$$\Rightarrow W\left(\sum c_i I_{(s_i,t_i]}\right) = \sum c_i \left(B_{t_i}-B_{s_i}\right)$$

$$W(f) = \int_0^\infty f(s)\,dB_s \quad \text{WIENER INTEGRAL}$$

CONTINUOUS GRFs $\quad X_s:\Omega\to\mathbb{R}^n$ $(s\in D)$ GRF on $(\Omega,\mathcal{A},P)$

Suppose $D\subseteq\mathbb{R}^d$ meas., and $s\mapsto X_s(\omega)$ continuous for (almost) every $\omega\in\Omega$

Lemma 17 1) $\mu=P\circ X^{-1}$ Gaussian measure on $E=C(D,\mathbb{R}^n)$

2) $m(s)=E[X_s]$ and $C(s,t)=\mathrm{Cov}(X_s,X_t)$ are continuous functions in s and t.

Proof $s_n\to s,\ t_n\to t \Rightarrow \overbrace{X_{s_n}\to X_s,\ X_{t_n}\to X_t}^{\text{jointly Gaussian}}$ a.s.

$$\overset{\text{Exercise}}{\Longrightarrow} E[X_{s_n}]\to E[X_s],\ \mathrm{Cov}(X_{s_n},X_{t_n})\to\mathrm{Cov}(X_s,X_t).\ \square$$

<u>Remark</u> $m(s) = m(\delta_s)$, $C(s,t) = C(\delta_s, \delta_t)$

determine a Gaussian measure on $C(D,\mathbb{R}^n)$ uniquely.

<u>COVARIANCE OPERATOR</u> $g, h \in L^2(D,\mathbb{R}^n) \Rightarrow$

$$\mathrm{Cov}\Big(\underbrace{\int_D g(s)\cdot X_s\,ds}, \underbrace{\int_D h(s)\cdot X_s\,ds}\Big) \overset{\text{bilin}}{=} \iint_{D\times D} g(s)\cdot C(s,t)\, h(t)\,dt$$

$$\mathrm{Cov}\big((g,X)_{L^2(D,\mathbb{R}^n)}, (h,X)_{L^2(D,\mathbb{R}^n)}\big) = (g, Kh)_{L^2(D,\mathbb{R}^n)}$$

$$(Kh)(s) := \int_D C(s,t)\,h(t)\,dt$$ <u>COVARIANCE OPERATOR W.R.T. L^2 METRIC</u>

$K: L^2(D,\mathbb{R}^n) \to L^2(D,\mathbb{R}^n)$ bounded symmetric non-negative

Moreover, K is compact if D is bounded.

<u>Lemma 18</u> (Mercer's Theorem) Suppose that D is bounded, and $C: D\times D \to \mathbb{R}^{n\times n}$ is symmetric, non-negative definite and continuous. Then K is a <u>compact</u> linear operator on $L^2(D,\mathbb{R}^n)$, and

$$C(s,t) = \sum_{k=1}^{\infty} \lambda_k\, e_k(s) \otimes e_k(t) \quad \text{uniformly on } D\times D$$

where $\{e_k\}$ is an ONB of eigenfunctions of K with eigenvalues $\lambda_k \downarrow 0$, e_k continuous for $\lambda_k \neq 0$.

Sketch of proof:

1) Image of $\{f \in L^2(D,\mathbb{R}^n) : \|f\|_{L^2} \leq 1\}$ under K is equicontinuous

$\Longrightarrow$ relatively compact by Arzéla-Ascoli

$\Longrightarrow$ K compact symm. lin. operator, non-neg.

Spectral Thm.
$\Longrightarrow$ $\exists$ ONB $\{e_k\}$ of eigenfunctions with eigenvalues $\lambda_k \geq 0$

$$Kf = \sum_k \lambda_k (f, e_k)\, e_k \quad \text{in } L^2 \quad \forall f \in L^2(D,\mathbb{R}^n)$$

2) $\lambda_k e_k(s) = \underbrace{\int C(s,t) e_k(t)\, dt}_{\text{continuous}}$

$\Rightarrow$ e_k contin. if $\lambda_k \neq 0$

3) Uniform convergence of $\sum \lambda_k e_k(s) e_k(t)$:

$$(*) \quad (f, Kf)_{L^2} = \sum \lambda_k (f, e_k)(e_k, f) \quad \forall f \in L^2$$

$$\begin{array}{ccc} \| & & \| \\ \iint f(s) C(s,t) f(t) & & \iint f(s) \left[\sum_k \lambda_k e_k(s) e_k(t)\right] f(t) \end{array}$$

$\Longrightarrow$ $\sum \lambda_k |e_k(s)|^2 \leq \sup C(\cdot,\cdot) \overset{\text{C.S.}}{=} \sup_{s} C(s,s)$

$\overset{C.S.}{\Longrightarrow}$ $\sum \lambda_k |e_k(s)|\,|e_k(t)| \leq \sup C \quad \forall s,t$

Weierstrass
$$\Rightarrow \quad \sum \lambda_k e_k(s) \otimes e_k(t) \text{ unif. conv. on } D\times D$$
$$\Rightarrow \quad \text{Limit} = C(s,t) \,.$$
$\square$

$(X_s)_{s\in D}$ centered continuous GRF with covariance $C(s,t)$,

$\nu = P\circ X^{-1}$ Law on $E = C(D,\mathbb{R}^n)$

THEOREM 19 Suppose that D is bounded and $\ker(K) = \{0\}$

Then ν is a standard normal distribution w.r.t.
$$(g,h)_H := (K^{-1/2} g, K^{-1/2} h)_{L^2(D,\mathbb{R}^n)} \,, \quad H = \mathrm{Dom}(K^{-1/2})$$

Rem. $(g,h)_H = (g, K^{-1} h)_{L^2(D,\mathbb{R}^n)}$ for $h \in \mathrm{Dom}(K^{-1})$

Proof To show: $\mathrm{Var}_\nu(\ell) = \|\ell\|^2_{H^*} \quad \forall \ell \in E^* \quad (*)$

$E \subset L^2(D,\mathbb{R}^n)$ dense $\Rightarrow L^2(D,\mathbb{R}^n)^* \subseteq E^*$ dense

$\Rightarrow$ suffices to show $(*)$ for $\ell \in L^2(D,\mathbb{R}^n)^* \cong L^2(D,\mathbb{R}^n)$

Hence let $\ell(x) = (h, x)_{L^2(D,\mathbb{R}^n)}$ with $h \in L^2(D,\mathbb{R}^n)$. Then:
$$\mathrm{Var}_\nu(\ell) = \mathrm{Var}[(h,X)_{L^2(D,\mathbb{R}^n)}] = (h, Kh)_{L^2} = \|Kh\|^2_H \quad (**)$$

Moreover, for $x \in H = \mathrm{Dom}(K^{-1/2})$,

$$\ell(x) = (\ell, x)_{L^2} = (K^{-1/2} K \ell, K^{-1/2} x)_{L^2} = (K\ell, x)_H .$$

Hence $K\ell$ is the element in H associated with ℓ via the Riesz isometry $H \cong H^*$, and, therefore, (*) holds by (**). □

<u>Consequence</u>: $\{e_k\}$ ONB of $L^2(D, \mathbb{R}^n)$ consisting of eigenfunctions of K with eigenvalues $\lambda_k > 0$

$\Rightarrow$ $\{\sqrt{\lambda_k}\, e_k\}$ ONB of H

$$(\sqrt{\lambda_k}\, e_k, \sqrt{\lambda_\ell}\, e_\ell)_H = \sqrt{\lambda_k \lambda_\ell}\, (K^{-1/2} e_k, K^{-1/2} e_\ell)_{L^2} = (e_k, e_\ell)_{L^2} = \delta_{k\ell}$$

<u>THEOREM 20</u> (Karhunen-Loève Expansion)

$D \subset \mathbb{R}^d$ bounded, $(X_s)_{s \in D}$ centered continuous GRF

$$\Rightarrow \quad X_s(\omega) = \sum_{\lambda_k \neq 0} \sqrt{\lambda_k}\, Z_k(\omega)\, e_k(s) \quad \text{a.s. } \underline{\text{uniformly}} \text{ on } D$$

where $\{e_k\}$ is ON eigenbasis of covariance operator on $L^2(D, \mathbb{R}^n)$, and $\{Z_k\}$ are i.i.d. $\sim N(0,1)$

<u>Rem.</u> $Z_k = \lambda_k^{-1/2} (X, e_k)_{L^2(D, \mathbb{R}^n)}$ $\Rightarrow$ Conv. in $L^2(D, \mathbb{R}^n)$ is obvious

Proof cf. below (Theorem is special case of a corresponding expansion for general Gaussian random variables with values in a Banach space E; the point is that the expansion converges strongly w.r.t. $\|\cdot\|_E$!)

Application of KL expansion: <u>Simulation of GRF</u> !

EXAMPLE: THE BROWNIAN SHEET

<u>Def.</u> A <u>Brownian sheet</u> is a continuous centered GRF $B_s : \Omega \to \mathbb{R}$, $s \in [0,1]^d$ (or $\mathbb{R}_+^d$), with covariance

$$C(s,t) = \prod_{i=1}^{d} \min(s_i, t_i)$$

Existence follows from Kolmogorov-Centsov Theorem, cf. below

<u>$d=1$</u>: Brownian sheet = standard Brownian motion

$C(s,t) = \min(s,t)$ Green's fction of $-\frac{d^2}{ds^2}$ with Dirichlet b.c. at 0 and Neumann b.c. at 1

$\Rightarrow$ K^{-1} = self-adj. realiz. of $-\frac{d^2}{ds^2}$ with Di./Neumann b.c.

$(g,h)_H = \int_0^1 g'(s)\cdot h'(s)\,ds$, $H = \{h \in C([0,1]) : h(0)=0, \exists h' \in L^2,$

$$\Longrightarrow \qquad e_k(s) = \sqrt{2}\,\sin\left(\frac{2k+1}{2}\pi s\right) \quad (k \in \mathbb{N})$$

ON Eigenbasis of K on $L^2([0,1])$, $\lambda_k = \left(\frac{2k+1}{2}\pi\right)^{-2}$

<u>$d>1$</u>: $$(Kh)(s) = \int_0^1 \cdots \int_0^1 \prod_{i=1}^{d} \min(s_i, t_i)\, h(t_1, \dots, t_d)\, dt_1 \cdots dt_d$$

i.e. $K_d = \bigotimes_{i=1}^{d} K_1$

$\Longrightarrow$ ON Eigenbasis given by

$$e_{k_1,\dots,k_d}(s_1,\dots,s_d) = \prod_{i=1}^{d} e_{k_i}(s_i), \quad k \in \mathbb{N}^d,$$

$$\lambda_{k_1,\dots,k_d} = \prod_{i=1}^{d} \left(\frac{2k_i+1}{2}\pi\right)^{-2}$$

$$K^{-1} = \bigotimes_{i=1}^{d} \underbrace{\left(-\frac{d^2}{ds_i^2}\right)}_{\text{self-adjoint with Dir./Neumann b.c.}} = (-1)^d \frac{\partial^{2d}}{\partial s_1^2 \cdots \partial s_d^2} + \text{b.c.}$$

$$(g,h)_H = \int_0^1 \cdots \int_0^1 \frac{\partial^d g}{\partial s_1 \cdots \partial s_d} \frac{\partial^d h}{\partial s_1 \cdots \partial s_d}\, ds_1 \cdots ds_d$$

$$H = \left\{ h \in C([0,1]^d) : h(0) = 0,\ \frac{\partial^d h}{\partial s_1 \cdots \partial s_d} \in L^2([0,1]^d) \right\}$$

Properties of Brownian sheet (Exercise):

(i) $s_i \mapsto B_{s_1,\dots,s_d} \sim \sqrt{s_1 \cdots s_{i-1} s_{i+1} \cdots s_d}\ \underbrace{\tilde{B}_{s_i}}_{\text{standard BM}}$

vanishes on axes!

(ii) $t \mapsto B_{t,\dots,t}$ has independent increments ($\Rightarrow$ martingale), not stationary

(iii) $d=2$, $t \to B_{\underbrace{e^t, e^{-t}}_{\text{hyperbola } s_1 s_2 = 1}}$ is an Ornstein-Uhlenbeck process

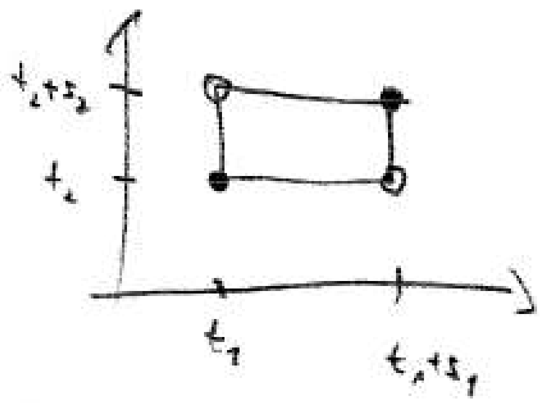
(iv) Scaling: $\left(\frac{1}{a_1 \cdots a_n} W_{a_1^2 s_1, \dots, a_n^2 s_n}\right) \sim \left(W_{s_1,\dots,s_n}\right)$

(v) Inversion: $\left(W_{s_1,\dots,s_n}\right) \sim \left(s_1 W_{\frac{1}{s_1}, s_2, \dots, s_n}\right) \sim \left(s_1 \cdots s_n W_{\frac{1}{s_1},\dots,\frac{1}{s_n}}\right)$

(vi) Translation invariance: (e.g. for $d=2$, $t_1, t_2 \in \mathbb{R}_+$)

$$\left(W_{t_1+\cdot,\, t_2+\cdot} - W_{t_1+\cdot,\, t_2}\right) - \left(W_{t_1,\, t_2+\cdot} - W_{t_1,t_2}\right) \sim W$$

independent of $\sigma(W_{u,v} : u \le t_1, v \le t_2)$



<u>Connection to white noise</u>: $(B_s)_{s \in [0,1]^d}$ Br. sheet

$\Rightarrow \exists!$ white noise W on $([0,1]^d, \mathcal{B}([0,1]^d), \lambda^d)$ s.t.

$$(*) \quad W\left(\prod_{i=1}^{d} (0, s_i] \right) = B_{s_1, \dots, s_d} \quad \forall s \in [0,1]^d$$

Conversely: W white noise w.r.t. λ^d

$\Rightarrow$ B defined by $(*)$ has properties of Brownian sheet except continuity, $\exists$ continuous modification, cf. below

2.7 Continuity of Random Fields

E sep. Banach space, $(\Omega, \mathcal{A}, P)$ prob. space

$X_s : \Omega \to E$ $(s \in [0,1]^d)$ random field

$\exists$ continuous modification $(\tilde{X}_s)$? Modulus of continuity?

$\omega(X, \delta) = \sup\{\|X_s - X_t\| : |s-t| \le \delta\}$

Ref. Adler: Random Fields & Geometry

Friz/Victoir: Rough paths, Appendix A $(d=1)$

An analytic ("pathwise") result: $x : [0,1]^d \to E$ measurable

THEOREM 21 (Garsia-Rodemich-Rumsey)

Let $\psi, p : [0,\infty) \to [0,\infty)$ continuous & strictly increasing with $\psi(0) = p(0) = 0$,
ψ convex $(\Rightarrow \lim_{r\to\infty} \psi(r) = \infty)$. If

$$C := \int_{[0,1]^d} \int_{[0,1]^d} \psi\left(\frac{\|x(s) - x(t)\|}{p(|s-t|/\sqrt{d})}\right) ds\, dt < \infty$$

then $\exists$ Lebesgue measure zero set N s.t.

$$(*) \quad \|x(s) - x(t)\| \le 8 \int_0^{|s-t|} \psi^{-1}\left(\frac{C}{r^{2d}}\right) dp(r) \quad \forall s,t \in [0,1]^d \setminus N$$

Explicit bound for modulus of continuity via integral criterion!

Remarks 1) If x is continuous then (*) holds for any s,t.

2) Assertion also holds if ψ is not convex, cf. e.g. Friz.

3) $[0,1]^d$ may be replaced by any other cube in $\mathbb{R}^d$.

4) Assertion holds with $|s-t|_\infty = \max_i |s_i - t_i|$ instead of $|s-t|/\sqrt{d}$

Example (Besov-Hölder embedding) $\psi(r) = r^q$, $p(r) = r^{\alpha/q}$

$$C = d^{\alpha/2} \int_{[0,1]^d}\int_{[0,1]^d} \frac{\|x(s)-x(t)\|^q}{|s-t|^\alpha}\, ds\, dt \qquad \left(\frac{\alpha-1}{q}, q\right)\text{-Besov norm}$$

$C < \infty$ for $q \geq 1$, $\alpha > 2d$ $\Longrightarrow$

$$\|x(s)-x(t)\| \leq 8\, C^{1/q} \frac{q}{\alpha-2d} |s-t|^{\frac{\alpha-2d}{q}} \text{ a.s. Hölder cont.}$$

$$\|x\|_{\text{Höl}\left(\frac{\alpha-2d}{q}\right)} \leq 8\, C^{1/q} \frac{q}{\alpha-2d} \|x\|_{\text{Besov}\left(\frac{\alpha-1}{q}, q\right)} \quad \forall \text{ cont. } x$$

Proof of Theorem 21

① W.l.o.g. $E = \mathbb{R}^1$.

general case: $\ell \in E'$ with $\|\ell\|_{E'} \leq 1 \Rightarrow |\ell(x(t)) - \ell(x(s))| \leq \|x(t)-x(s)\|$

$\Rightarrow$ $\ell \circ x$ satisfies assumption with same constant C

$\Rightarrow$ (*) holds for $\ell \circ x$ $\Rightarrow$ (*) holds for $\|x(s)-x(t)\| = \sup_{\|\ell\|_{E'} \leq 1} |\ell(x(s)) - \ell(x(t))|$

(2) Notation: $Q \subseteq [0,1]^d$ cube, $e(Q)$ edge length

$$|s-t| \le \sqrt{d}\, e(Q) \quad \forall s,t \in Q$$

$$\overset{\rho \text{ incr.}}{\Longrightarrow} \quad \int_Q \int_Q \Psi\left(\frac{|x(s)-x(t)|}{\rho(e(Q))}\right) ds\, dt \le C \qquad (**)$$

$$x_Q := \fint_Q x(s)\, ds = \frac{1}{\mathrm{vol}(Q)} \int_Q x(s)\, ds \quad \text{average}$$

(3) Averaged bound: $\widehat{Q} \subseteq Q \subseteq [0,1]^d$ cubes

$$\Rightarrow \quad |x_Q - x_{\widehat{Q}}| \le 4 \int_0^{e(Q)} \Psi^{-1}\left(\frac{C}{r^{2d}}\right) d\rho(r)$$

Proof by "chaining argument": $Q = Q_0 \supset Q_1 \supset \dots \supset Q_n = \widehat{Q}$ cubes s.t. $e_k := e(Q_k)$ satisfies $\rho(e_{k+1}) = \frac{1}{2}\rho(e_k)$ for $k \ge 1$, $\rho(e_1) \ge \frac{1}{2}\rho(e_0)$;

enough to show: $|x_{Q_k} - x_{Q_{k-1}}| \le 4 \int_{e_{k+1}}^{e_k} \Psi^{-1}\left(\frac{C}{r^{2d}}\right) d\rho(r)$

$$\Psi\left(\frac{\overbrace{|x_{Q_k} - x_{Q_{k-1}}|}^{= \fint_{Q_k}\fint_{Q_{k-1}} (x(s)-x(t))\, ds\, dt}}{\rho(e_{k-1})}\right) \overset{\Psi \text{ convex}}{\le} \fint_{Q_k} \fint_{Q_{k-1}} \Psi\left(\frac{|x(s)-x(t)|}{\rho(e_{k-1})}\right) ds\, dt$$

$$\overset{(**)}{\le} \frac{C}{V_k V_{k-1}}, \quad V_k := \mathrm{Vol}(Q_k)$$

ψ incr.

$$\Longrightarrow \quad |X_{Q_k} - X_{Q_{k-1}}| \le \underbrace{p(e_{k-1})}_{\le 4(p(e_k)-p(e_{k+1}))} \overset{\text{increasing}}{\psi^{-1}}\Big(\frac{C}{\underbrace{V_k V_{k-1}}_{\le C/e_k^{2d}}} \Big)$$

$$\le 4 \int_{e_{k+1}}^{e_k} \psi^{-1}\Big(\frac{C}{r^{2d}}\Big)\, dp(r) \qquad \forall k \ge 1$$

(4) <u>Pointwise bound:</u>

$$\mathcal{D}_n := \Big\{ \prod_{i=1}^{d} \big[(k_i-1)2^{-n}, k_i 2^{-n}\big) : k_1, \dots, k_d \in \{1, \dots, 2^n\} \Big\} \quad \text{dyadic cubes}$$

Martingale conv. theorem: $E[X \mid \sigma(\mathcal{D}_n)] \to X$ a.e.

i.e. $\exists$ null set N s.t. for $s \in N^c$,

$$X(s) = \lim_{n\to\infty} X_{Q_n(s)}, \quad Q_n(s) \in \mathcal{D}_n \text{ with } s \in Q_n(s)$$

$s,t \in N^c$

$$\Longrightarrow \quad |X(s) - X(t)| = \lim_{n\to\infty} \underbrace{|X_{Q_n(s)} - X_{Q_n(t)}|}_{\substack{\le |X_{Q_n(s)} - X_Q| + |X_Q - X_{Q_n(t)}| \\ \le 8 \int_0^{e(Q)} \psi^{-1}\left(\frac{C}{r^{2d}}\right) dp(r)}} \le 8 \int_0^{|s-t|} \psi^{-1}\Big(\frac{C}{r^{2d}}\Big)\, dp(r)$$

with $Q \supseteq Q_n(s) \cup Q_n(t)$ cube, $e(Q) \le |s-t| + 2\cdot 2^{-n}$ $\square$

<u>Rem.</u> More precise formulation of GRR Theorem: $x : [0,1]^d \to E$ meas.,

$\Psi, p : [0,\infty) \to [0,\infty)$ cont & strictly incr with $\Psi(0) = p(0) = 0$,

Ψ convex. Suppose that

$$C := \int_{[0,1]^d}\int_{[0,1]^d} \Psi\left(\frac{\|x(s)-x(t)\|}{p(|s-t|/\sqrt{d})}\right) ds\, dt < \infty \quad \text{, and}$$

$$\varrho(\delta) := \int_0^\delta \Psi^{-1}\left(\frac{C}{r^{2d}}\right) dp(r) < \infty \quad \text{for some } \delta > 0.$$

Then

$$\tilde{x}(s) := \lim_{n\to\infty} x_{Q_n(s)} \quad \text{exists for } \underline{\text{every}}\ s \in [0,1]^d,$$

$\tilde{x} = x$ a.e, and $\tilde{x}$ is continuous with <u>modulus of continuity</u>

$$\omega(\tilde{x},\delta) := \sup_{|s-t|\leq\delta} \|\tilde{x}(s) - \tilde{x}(t)\| \leq \varrho(\delta) \quad \forall \delta > 0.$$

Proof: Exercise

Remark/Exercise: More precisely the proof shows: If $\int_0^\delta \psi^{-1}\left(\frac{C}{r^{2d}}\right) dp(r) < \infty$ for some $\delta > 0$ then $\tilde{X}_s := \lim_{n\to\infty} X_{Q_n(s)}$ exists for all s, $\tilde{X}$ is continuous with $\tilde{X} = X$ a.s. and satisfies ...

<u>Application of Garsia-Rodemich-Rumsey to random fields:</u>

$X_s : \Omega \to E \quad (s \in [0,1]^d)$ random field on $(\Omega, \mathcal{A}, P)$

Suppose that

$$\int_{[0,1]^d} \int_{[0,1]^d} E\left[\psi\left(\frac{\|X_s - X_t\|}{p(|s-t|/\sqrt{d})}\right)\right] ds\, dt < \infty$$

Then we obtain a bound for the modulus of continuity of the modification $\tilde{X}_s := \lim_{n\to\infty} X_{Q_n(s)}$:

$$\omega(\tilde{X}, \delta) := \sup_{|s-t| \le \delta} \|\tilde{X}_s - \tilde{X}_t\| \le 8 \int_0^\delta \psi^{-1}\left(\frac{C}{r^{2d}}\right) dp(r)$$

with $C \in \mathcal{L}^1(\Omega, \mathcal{A}, P)$

Possible choices for ψ, p:

(i) $\psi(r) = r^q$, $p(r) = r^\gamma$ $\rightsquigarrow$ Kolmogorov-Centsov Hölder continuity

(ii) $\psi(r) = \exp(\varepsilon r^2)$ $\rightsquigarrow$ Improved estimate for Gaussian random fields

COROLLARY 22 (Kolmogorov - Centsov)

$X_s : \Omega \to E$ $(s \in [0,1]^d)$ random field. If there exist constants $q \geq 1$, $\varepsilon > 0$, $K \in (0,\infty)$ s.t

$$E\left[\|X_s - X_t\|^q \right] \leq K \cdot |s-t|^{d+\varepsilon} \quad \forall s,t \in [0,1]^d$$

then (X_s) has a continuous modification $(\tilde{X}_s)$ (i.e. $P[\tilde{X}_s = X_s] = 1 \ \forall s$) satisfying

$$\equiv \left\| \sup_{s \neq t} \frac{\|\tilde{X}_s - \tilde{X}_t\|}{|s-t|^\gamma} \right\|_{L^q(\Omega,\mathcal{A},P)} < \infty \quad \text{for any } \gamma \in \left(0, \frac{\varepsilon}{q}\right)$$

<u>Proof</u> Expectation of Besov norm:

$$\underbrace{E\left[\iint_{[0,1]^d [0,1]^d} \frac{\|X_s - X_t\|^q}{|s-t|^\alpha} \, ds \, dt \right]}\overset{\text{Fubini}}{\leq} K \cdot \iint \frac{|s-t|^{d+\varepsilon}}{|s-t|^\alpha} \, ds \, dt$$

$$\leq K \cdot \int_{[0,1]^d} |u|^{d+\varepsilon-\alpha} \, du \leq \text{Const.} \cdot \int_0^{\sqrt{d}} r^{d+\varepsilon-\alpha} r^{d-1} \, dr$$

$$< \infty \quad \text{for any } \alpha < 2d + \varepsilon$$

In this case:

$$E[\sqcup] < \infty \implies \sqcup < \infty \text{ a.s.}$$

Besov-Hölder embedding

$$\implies \tilde{X}_s := \begin{cases} \lim_{n\to\infty} X_{a_n(s)} & \text{if limit exists} \\ 0 & \text{otherwise} \end{cases} \quad \text{for } \alpha > 2d$$

with $\gamma = \frac{\alpha - 2d}{q}$ $\square$

Remark/Exercise Refinement: $\exists \gamma > 0$, r.v. $Y \in L^q(\ldots)$

$$|X_t - X_s| \leq Y \, |t-s|^{\varepsilon/q} \left(\log \frac{\gamma}{|t-s|}\right)^{2/q}$$

Application to tightness of probability measures on $C([0,1]^d, E)$:

$(X_s^{(i)})_{s\in[0,1]^d}$, $i \in I$, family of random fields, continuous

$\mu^{(i)} := P\circ(X^{(i)})^{-1}$ law of $X^{(i)}$ on $C([0,1]^d, E)$

COROLLARY 23 Suppose that $\sup_i P[X_0^{(i)} \geq \lambda] \to 0$ as $\lambda \to \infty$, and

$$K := \sup_{i\in I} \sup_{s\neq t} \frac{E[\|X_s^{(i)} - X_t^{(i)}\|^q]}{|s-t|^{d+\varepsilon}} < \infty \quad \text{for some } q \geq 1, \varepsilon > 0.$$

Then $\{\mu^{(i)} : i \in I\}$ is tight, i.e.,

(*) $\forall \delta > 0 \; \exists A \subseteq C([0,1]^d, E)$ rel. compact: $\sup_{i\in I} \mu^{(i)}(A^c) < \delta$

Consequence (by Prokhorov's Theorem):

$\{\mu^{(i)}\}$ is relatively compact w.r.t. weak convergence,
i.e., every sequence $(\mu^{(i_k)})_{k\in\mathbb{N}}$ has a weakly conv. subsequence.

Proof: Similarly to the proof of Cor. 22, we obtain for $\gamma\in(0,\frac{\varepsilon}{q})$:

$$\sup_{i\in I} E\left[\|X^{(i)}\|^q_{Höl(\gamma)}\right] < \infty .$$

Hence by Markov's inequality,

$$\sup_{i\in I}\mu^{(i)}\left(\{x\in C([0,1]^d,E) : \|x\|_{Höl(\gamma)}\geq c\}\right)\leq \text{const.}\cdot c^{-q} \underset{\uparrow\, c \text{ large}}{<} \frac{\delta}{2}$$

$\Rightarrow$ (*) holds with $A=\{x : \|x\|_{Höl(\gamma)}<c, \|x(0)\|<\lambda\}$, λ large.

A is rel. cp. in $C([0,1]^d,E)$ by Arzelà-Ascoli. $\square$

Rem. $\{\mu^{(i)} : i\in I\}$ is even tight in $C_\gamma([0,1]^d,E)$ for any $\gamma<\varepsilon/q$.
Proof: Chose $\gamma<\gamma'<\varepsilon/q$ $\Rightarrow \{x : \|x\|_{Höl(\gamma')}<c, \|x(0)\|<\lambda\}$ rel. cp. in C_γ.

Continuity of Gaussian Random fields

$(X_s)_{s\in[0,1]^d}$ GRF on $(\Omega, \mathcal{A}, P)$

Fernique's Theorem: $\exists$ universal constants $\varepsilon>0$, $C_0<\infty$ s.t.

$$E\left[\exp\left(\varepsilon \|Z\|^2 / E[\|Z\|^2]\right)\right] \leq C_0$$

for any Gaussian random variable $Z: \Omega \to E$.

In particular: $\psi(r) := \exp(\varepsilon r^2) - 1$

$$\Rightarrow \quad E\left[\psi\left(\frac{\|X_s - X_t\|}{p(|s-t|/\sqrt{d})}\right)\right] \leq C_0 \quad \forall s \neq t \quad (*)$$

where $p(r) := \sup\limits_{|s-t| \leq r\sqrt{d}} E\left[\|X_s - X_t\|^2\right]^{1/2}$ "mean square modulus of continuity"

COROLLARY 24 (Modulus of continuity for GRFs). If $\lim\limits_{r\to 0} p(r) = 0$ then
$\exists C \in \mathcal{L}^1(\Omega, \mathcal{A}, P)$:

$$\omega(\tilde{X}, \delta) \leq \frac{8}{\sqrt{\varepsilon}} \int_0^\delta \sqrt{\log\left(1 + \frac{C}{r^{2d}}\right)}\, dp(r) \qquad \forall \delta > 0$$

<u>Proof</u>: Apply GRR with Ψ, p as above, $\Psi^{-1}(t) = \sqrt{\frac{\log(1+t)}{\varepsilon}}$

$$C = \iint \Psi\left(\frac{\|X_s - X_t\|}{p(|s-t|/\sqrt{d})}\right) ds\, dt \quad \text{satisfies } E[C] = C_0 < \infty.$$

$\square$

<u>Example</u> 1) $p(t) \leq \dfrac{c}{\left(\log \frac{1}{t}\right)^{\beta}} \quad \beta > \frac{1}{2}$

$\Rightarrow \exists$ cont. modif.

<u>Example</u> 1) Br. sheet

$$E\left[\|X_s - X_t\|^2\right] = C(s,s) + C(t,t) - 2C(s,t)$$
$$\leq 2d\,|t-s|$$

$p(t) \leq \sqrt{2d|t-s|}$

$\rightarrow$ Hölder β for $\beta < \frac{1}{2}$

2) Fr. Br. motion $C(s,t) = \dfrac{s^{\alpha} + t^{\alpha} - |s-t|^{\alpha}}{2}$, $\alpha \in (0,2)$

$E\left[\|X_s - X_t\|^2\right] = |s-t|^{\alpha}$ $\quad p(t) \leq \sqrt{d}\,|t|^{\alpha/2}$

$\rightarrow$ Hölder β for $\beta < \alpha$

2.8. Cameron-Martin space (Reproducing Kernel Hilbert space)

μ centered Gaussian measure on sep. Banach space E

$$C(\ell,\tilde{\ell}) = \mathrm{Cov}_\mu(\ell,\tilde{\ell}) = \int \ell\tilde{\ell}\,d\mu, \quad \ell,\tilde{\ell}\in E^*$$

Representation as standard normal distribution?

Def. (Cameron-Martin space)

$$H(\mu) := \{h\in E : \|h\|_{H(\mu)} < \infty\}, \text{ where}$$

$$\|h\|_{H(\mu)} := \sup_{\substack{\ell\in E^*\\ C(\ell,\ell)\le 1}} \ell(h) = \sup_{\substack{\ell\in E^*\\ \|\ell\|_{L^2(\mu)}\le 1}} \ell(h)$$

Rem. (E,H,μ) abstract Wiener space $\Rightarrow H(\mu) = H$

Proof: (i) $h\in H \Rightarrow \ell(h) \underset{\substack{\uparrow\\ g \text{ assoc. to } \ell\\ \text{via Riesz isometry}\\ H\cong H^*}}{=} (g,h)_H \le \|g\|_H\,\|h\|_H = \|\ell\|_{H^*}\|h\|_H$

$= C(\ell,\ell)^{1/2}\|h\|_H \quad \forall \ell\in E^*$

$\Rightarrow \|h\|_{H(\mu)} \le \|h\|_H < \infty \Rightarrow h\in H(\mu)$

(ii) $h\in H(\mu) \Rightarrow \ell(h) \le \underbrace{C(\ell,\ell)^{1/2}}_{=\|\ell\|_{H^*}}\|h\|_{H(\mu)} \quad \forall \ell\in E^*$

$\overset{E^*\subseteq H^* \text{ dense}}{\Longrightarrow} h\in (H^*)^* = H, \quad \|h\|_H \le \|h\|_{H(\mu)}$ $\square$

EXAMPLES

1) $E = \mathbb{R}^n$, $C_{ij} = \mathrm{Cov}_\mu(x_i, x_j)$, $E^* \cong \mathbb{R}^n$

$$\Rightarrow \quad \|h\|_{H(\mu)} = \sup_{\substack{x \in \mathbb{R}^n \\ x \cdot Cx \leq 1}} x \cdot h = \begin{cases} h \cdot C^{-1} h & \text{if } h \perp \ker C \\ +\infty & \text{otherwise} \end{cases}$$

$$H(\mu) = (\ker C)^\perp \subseteq \mathbb{R}^n \qquad (H(\mu) = \mathbb{R}^n \text{ if non-degenerate})$$

is a Hilbert space with inner product

$$(g, h)_{H(\mu)} = g \cdot C^{-1} h \,, \qquad C^{-1} \text{ inverse on } H(\mu)$$

2) $E = C(D, \mathbb{R}^n)$, $D \subset \mathbb{R}^d$ compact,

μ = distribution of GRF with covariance $C(s,t)$.

$$(Kh)(s) = \int_D C(s,t)\, h(t)\, dt$$

Suppose $\ker(K) = \{0\}$. Then by Theorem 19 & Remark 8:

$$H(\mu) = \mathrm{Dom}(K^{-1/2}) \qquad \text{Hilbert space}$$

$$(g, h)_{H(\mu)} = (K^{-1/2} g, K^{-1/2} h)_{L^2(D,\mathbb{R}^n)} \overset{\text{if } h \in \mathrm{Dom}(K^{-1})}{\downarrow} (g, K^{-1} h)_{L^2(D,\mathbb{R}^n)}$$

e.g. Brownian sheet: $H(\mu) = H = \{ h \in C([0,1]^d) : h(0) = 0,\ \frac{\partial^d h}{\partial s_1 \cdots \partial s_d} \in L^2([0,1]^d) \}$ (2.49a)

Reproducing kernel property: $\text{Span}\{ C(s,\cdot) : s \in D \} \subseteq H(\mu)$ dense

$$(h, C(s,\cdot))_{H(\mu)} = h(s) \quad \forall h \in H(\mu),\ s \in D$$

" $H(\mu)$ is reproducing kernel Hilbert space of C "

Proof (h-1): $g \in L^2(D) \Rightarrow Kg \in \text{Dom}(K^{-1}) \subseteq H(\mu)$

$$(h, Kg)_{H(\mu)} = (h, g)_{L^2(D)} \quad \forall h \in H(\mu)$$

Now choose Dirac sequence $g_k \to \delta_s$

$$\| K(g_k - g_m) \|^2_{H(\mu)} = (K(g_k - g_m), g_k - g_m)_{L^2}$$

$$= \underbrace{(Kg_k, g_k)_{L^2}}_{} + \underbrace{(Kg_m, g_m)_{L^2}}_{= \iint g_k(r) C(r,t) g_k(t) \to C(s,s)} - 2 \underbrace{(Kg_k, g_m)_{L^2}}_{} \xrightarrow{k,m \to \infty} 0$$

$\Rightarrow Kg_k$ Cauchy in $H(\mu) \Rightarrow C(s,\cdot) = \lim_{k\to\infty} Kg_k \in H(\mu)$

$$(h, C(s,\cdot))_{H(\mu)} = \lim_{k\to\infty} \underbrace{(h, Kg_k)_{H(\mu)}}_{= (h, g_k)_{L^2}} = h(s)$$

$\lrcorner$

Gaussian Hilbert space & White noise isometry

IDEA: $h \in H(\mu) \rightsquigarrow X_h(\omega) := (h,\omega)_{H(\mu)} \sim N(0, \|h\|^2_{H(\mu)})$

PROBLEM: Not defined for $\omega \notin H(\mu)$.

If $(h,\cdot)_{H(\mu)}$ extends to cont lin. functional $\ell \in E^*$ then $X_h := \ell$.

Definition of X_h for general $h \in H(\mu)$ via isometry:

$$\ell, \tilde{\ell} \in E^*: \ \ell \sim \tilde{\ell} :\Leftrightarrow \ell = \tilde{\ell} \ \mu\text{-a.s.} \Leftrightarrow C(\ell-\tilde{\ell}, \ell-\tilde{\ell}) = 0$$

$$G(\mu) := \overline{E^*/\sim} \subseteq L^2(\mu)$$

is a Hilbert space consisting of Gaussian random variables

"Gaussian Hilbert space"

Remarks 1) $G(\mu)$ is completion of $E^*/\ker C$ w.r.t. norm $C(\cdot,\cdot)^{1/2}$.

2) $\ell \in E^* \xrightarrow{\text{Riesz isometry}} \exists!\, h \in H(\mu): \ \ell(g) = (h,g)_{H(\mu)} \ \forall g \in H(\mu)$

$$\ell \in E^* \underset{\substack{\text{not 1-1}\\ \text{in general}}}{\longrightarrow} H(\mu)^* \cong H(\mu) \ni i(\ell) := h$$

3) By definition, $\|h\|_{H(\mu)} = \sup_{\|\ell\|_{G(\mu)} \leq 1} \ell(h) = \| \ell \mapsto \ell(h) \|_{G(\mu)^*}$.

THEOREM 25 (White noise isometry)

$$\begin{array}{ccc} E^*/\sim & \subseteq & H(\mu)^* \\ \cap & & \cong \text{ Riesz} \\ G(\mu) & \underset{\text{white noise isometry}}{\cong} & H(\mu) \end{array}$$

1) Let $h \in E$. Then:

$$h \in H(\mu) \iff \exists X_h \in G(\mu) : \forall \ell \in E^* \quad \ell(h) = (\ell, X_h)_{L^2(\mu)}$$

In this case, X_h is uniquely determined.

2) The map $h \mapsto X_h$ is an isometry.

$$H(\mu) \to G(\mu) \subseteq L^2(\mu)$$

In particular, $H(\mu)$ is a Hilbert space with inner product

$$(*) \quad (g, h)_{H(\mu)} = (X_g, X_h)_{L^2(\mu)}$$

3) Let $i : E^* \to H(\mu)$ be the Riesz isometry restricted to E^*.

Then $$X_{i(\ell)} = \ell \quad \text{for any } \ell \in E^*$$

Rem. $(X_h)_{h \in H(\mu)}$ is a "generalized white noise", i.e., it is a centered Gaussian process s.t. $h \mapsto X_h$ is a.s. linear, and

$$\mathrm{Cov}(X_g, X_h) = (g, h)_{H(\mu)}$$

Proof of Theorem 25

1), 2) $h \in H(\mu) \overset{\text{Def.}}{\iff} \ell(h) \leq \text{const.} \, \|\ell\|_{L^2(\mu)} \quad \forall \ell \in E^*$

In this case: $\exists!\, X_h \in \overline{E^*/\sim} = G(\mu) \subset L^2(\mu)$: $\ell(h) = (\ell, X_h)_{L^2} \; \forall \ell \in E^*$,

$$\|X_h\|_{L^2(\mu)} = \|(\ell \mapsto \ell(h))\|_{G(\mu)^*} \overset{\text{Def.}}{=} \|h\|_{H(\mu)}$$

3) $\ell \in E^*$, $h \in H(\mu)$ assoc. to $\ell|_{H(\mu)}$ via Riesz isometry

$$\Rightarrow (X_h, X_g)_{L^2(\mu)} \overset{(*)}{=} (h, g)_{H(\mu)} = \ell(g) = (\ell, X_g)_{L^2(\mu)} \quad \forall g \in H(\mu)$$

$$\Rightarrow (X_h - \ell, Y)_{L^2(\mu)} = 0 \quad \forall Y \in G(\mu) \Rightarrow X_h = \ell \text{ a.s.} \quad \square$$

In particular: $\ell = X_{i(\ell)} \sim N(0, \|i(\ell)\|^2_{H(\mu)}) \; \forall \ell \in E^*$, i.e., μ is $(\cdot,\cdot)_{H(\mu)}$ standard normal

EXAMPLES

1) $E = \mathbb{R}^n$, $H(\mu) = (\ker C)^\perp$, $(g, h)_{H(\mu)} = g \cdot \underset{\substack{\uparrow \\ \left(\text{inverse of } C \text{ on } H(\mu)\right)}}{C^{-1}} h$

$$h \in H(\mu) \xleftrightarrow{\text{Riesz isometry}} \ell(\omega) = \omega \cdot C^{-1} h \, , \quad \ell \in E^*$$

Hence: $\boxed{X_h(\omega) = \underbrace{(C^{-1} h) \cdot \omega}_{= (h, \omega)_{H(\mu)} \text{ for } \omega \in H(\mu)} \quad (\omega \in E)}$

2) <u>Classical Wiener space:</u> Let $h \in H$ = classical Cam. Martin space,

$$(h,g)_H = \int_0^1 h' g' = \int_0^1 h' \cdot dg \quad \forall g \in H$$

If h' has bounded variation then $(h,\cdot)_H \in H^*$ extends to continuous linear functional

$$l(\omega) = \int_0^1 h' \cdot d\omega \quad \text{on } E.$$

Hence:

$$\boxed{X_h(\omega) = \int_0^1 h' \, d\omega}$$ <u>stochastic integral of Wiener</u>

defined as unique isometric extension of $h \mapsto \int_0^1 h' d\omega \in L^2(\mu)$ from $\{h \in H : h' \text{ has b.v.}\}$ to H.

3) <u>Classical white noise:</u> $H(\mu) = L^2(\mathbb{R}^d)$

$$(h,g)_H = \int_{\mathbb{R}^d} h \, g \qquad \forall g \in H$$

If $h \in C_0^\infty(\mathbb{R}^d)$ then $(h,\cdot)$ extends to continuous linear functional $\langle h,\cdot\rangle$ on $C_0^\infty(\mathbb{R}^d)'$ (distribution).

$$\boxed{X_h = \langle h, \cdot \rangle \in L^2(\mu)} \text{ for } h \in C_0^\infty(\mathbb{R}^d),$$

defined by isometric extension for any $h \in L^2(\mathbb{R}^d)$.

Karhunen-Loève Expansion

— $H(\mu) \cong G(\mu) \subseteq L^2(E, \mu)$

$\Rightarrow$ $H(\mu)$ is separable Hilbert space

THEOREM 26 Let $\{e_n : n \in \mathbb{N}\}$ be an arbitrary ONB of the Cameron-Martin space $H(\mu)$. Then

$$\omega = \sum_{n=1}^{\infty} X_{e_n}(\omega)\, e_n \quad \text{a.s. w.r.t. } \|\cdot\|_E ,$$

and in $L^p(E \to E; \mu)$ for any $p \in [1, \infty)$.

Remark X_{e_n} $(n \in \mathbb{N})$ independent $\sim N(0,1)$ w.r.t. μ

Proof $\{e_n\}$ ONB of $H(\mu)$ $\Rightarrow$ $\{X_{e_n}\}$ ONB of $G(\mu) \cong \overline{E^*}^{\mu}$

$$\Rightarrow \quad \ell(\omega) = \sum_{n=1}^{\infty} \underbrace{(\ell, X_{e_n})_{L^2(\mu)}}_{= \ell(e_n) \text{ by def. of } X_{e_n}} X_{e_n}(\omega) \quad \text{in } L^2(\mu) \quad \forall \ell \in E^*$$

$$\Rightarrow \quad \ell\Big(\sum_{n=1}^{k} X_{e_n}(\omega)\, e_n\Big) \xrightarrow{k \uparrow \infty} \ell(\omega) \quad \text{in prob.} \quad \forall \ell \in E^*$$

$$\Rightarrow \sum_{n=1}^{k} X_{e_n}(\omega)\, e_n \longrightarrow l(\omega) \quad \text{a.s. w.r.t. } \|\cdot\|_E$$

$\uparrow$

by Theorem of Itô-Nisio on sums of independent symmetric random variables with values in Banach spaces, cf. below. $\square$

<u>COROLLARY 27</u> $Y: \Omega \to E$ Gaussian r.v. on $(\Omega, \mathcal{A}, P)$ with law $\mu = P\circ Y^{-1}$, $\{e_n : n \in \mathbb{N}\}$ ONB of $H(\mu)$

$$\Rightarrow \boxed{Y = \sum_{n=1}^{\infty} Z_n e_n \quad \text{a.s. in } E \text{ and in } L^p(\Omega \to E, P)}$$

with $Z_n := X_{e_n} \circ Y$ i.i.d. $\sim N(0,1)$.

<u>EXAMPLE</u> (Proof of Theorem 20) $(Y_s)_{s\in D}$ GRF

$e_n := \sqrt{\lambda_n}\, \tilde{e}_n$ where $\{\tilde{e}_n\}$ is ON eigenbasis of cov. op. on $L^2(D, \mathbb{R}^n)$

$$\Rightarrow \quad Y_s = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\, Z_n\, e_n(s) \quad \text{with } Z_n \text{ i.i.d. } \sim N(0,1)$$

$\uparrow$

$\{e_n\}$ ONB of $H(\mu)$

2.8.1. The Theorem of Itô and Nisio

Ref.: Jan van Neerven: Stoch. evolution equations, Lecture notes on homepage.

$(\Omega, \mathcal{F}, P)$ probability space, E sep. Banach space

Def. $X: \Omega \to E$ is called <u>symmetric</u> iff $X \sim -X$ w.r.t. P

Lemma 28 $X, Y: \Omega \to E$ independent, X symmetric

$$\Rightarrow E[\|X\|^p] \leq E[\|X+Y\|^p] \quad \forall p \in [1, \infty)$$

Proof $X+Y \sim -X+Y$ by symmetry + independence

$$\Rightarrow \|X\|_{L^p} = \frac{1}{2}\|X+Y+X-Y\|_{L^p} \leq \frac{1}{2}\|X+Y\|_{L^p} + \frac{1}{2}\|X-Y\|_{L^p} = \|X-Y\|_{L^p}.$$

$\square$

$S_n = X_1 + \dots + X_n$ Random walk

$X_i : \Omega \to E$ independent & symmetric

__Lemma 29__ (Lévy's inequality)

$$P\Big[\max_{1 \le k \le n} \|S_k\| \geq r\Big] \le 2 P\big[\|S_n\| \geq r\big] \quad \forall n \in \mathbb{N}, r \geq 0$$

__Proof__ Fix $k \in \mathbb{N}$. Reflection principle:

$$\tilde{S}_n^{(k)} := \begin{cases} S_n & \text{for } n \le k \\ S_k - (S_n - S_k) & \text{for } n \geq k \end{cases}$$

$$\Rightarrow \big(\tilde{S}_n^{(k)}\big)_{n \geq 0} \sim \big(S_n\big)_{n \geq 0} \quad \Big(\text{since } S_n = \underbrace{S_k}_{\text{independent}} + \underbrace{(S_n - S_k)}_{\text{symmetric}} \text{ for } n \geq k\Big)$$

$$\Rightarrow P\big[\overbrace{\|S_1\| < r, \dots, \|S_{k-1}\| < r, \|S_k\| \geq r}^{=: A_k}\big]$$

$S_k = \frac{1}{2}(S_n + \tilde{S}_n) \to$
$$\le P\big[A_k \cap \{\|S_n\| \geq r\}\big] + P\big[A_k \cap \{\|\tilde{S}_n^{(k)}\| \geq r\}\big]$$
$$= 2 P\big[A_k \cap \{\|S_n\| \geq r\}\big]$$

$$\Rightarrow P\Big[\max_{k \le n} \|S_k\| \geq r\Big] = \sum_{k=1}^{n} P[A_k] \le 2 P\big[\|S_n\| \geq r\big]. \quad \square$$

THEOREM 30 (Itô-Nisio)

$X_n : \Omega \to E$ $(n \in \mathbb{N})$ indep. symm. random variables,

$S_n = \sum_{k=1}^{n} X_k$, $S : \Omega \to E$ random variable.

Then the following assertions are equivalent:

(i) $\forall \ell \in E^*$: $\ell(S_n) \to \ell(S)$ a.s.

(ii) $\forall \ell \in E^*$: $\ell(S_n) \to \ell(S)$ in prob.

(iii) $\|S_n - S\| \to 0$ a.s.

(iv) $\|S_n - S\| \to 0$ in prob.

In this case, if $E[\|S\|^p] < \infty$ for some $p \in [1,\infty)$

then also
$$E[\|S_n - S\|^p] \to 0 .$$

<u>Remark</u> Special case of Martingale Convergence Theorem for Banach space valued martingales, cf. [Neveu].

<u>Proof</u> (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) ✓

(ii) $\Rightarrow$ (iv):

(1) $\{S_k : k \in \mathbb{N}\}$ is <u>tight</u>, i.e.,

$\forall \varepsilon > 0 \;\exists L \subset E$ compact: $\sup_k P[S_k \notin L] < \varepsilon$

$\tilde{S}^{(k)} := S_k - (S - S_k) = 2S_k - S$

$\tilde{S}^{(k)} \sim S$ since $\ell(\tilde{S}^{(k)}) \xleftarrow{P} \ell(\tilde{S}^{(k)}_n) \sim \ell(S_n) \xrightarrow{P} \ell(S) \;\forall \ell \in E^*$.

Choose $K \subset E$ cp. s.t. $P[S \notin K] < \varepsilon/2$.

$S_k = \frac{1}{2}(S + \tilde{S}^{(k)}) \quad \forall k \in \mathbb{N}$

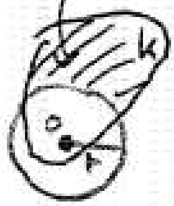
$\Rightarrow \; P[S_k \notin \frac{1}{2}(K - K)] \leq P[S \notin K] + P[\tilde{S}^{(k)} \notin K]$

$= 2\, P[S \notin K] < \varepsilon \quad \forall k \in \mathbb{N}$

(2) $\{S_k - S : k \in \mathbb{N}\}$ tight by (1), $\ell(S_k - S) \xrightarrow{P} 0 \;\forall \ell \in E^*$

Suppose $\|S_k - S\| \not\xrightarrow{P} 0$

prob. $> \varepsilon/2 \ \forall n$



$\Rightarrow$ $\exists$ subsequence, $r, \varepsilon > 0$, K cp.:

$$P[\|S_{k_n} - S\| > r] \geq \varepsilon \quad \forall n$$

$$P[S_{k_n} - S \notin K] \leq \varepsilon/2 \quad \forall n$$

$\Rightarrow$ (cover K by finitely many balls of radius $r/2$)

$\exists$ ball B s.t. $0 \notin B$, $\inf_n P[S_{k_n} - S \in B] > 0$

$\overset{\text{Hahn-Banach}}{\underset{\ell \geq 1 \text{ on } B}{\Longrightarrow}}$ $\exists\, \ell \in E^*$: $\inf_n P[\ell(S_{k_n} - S) \geq 1] > 0$ $\lightning$

(iv) $\Rightarrow$ (iii): via Lévy's inequality:

$S_k \xrightarrow{P} S \Rightarrow \exists$ subseq. $S_{k_n} \to S$ a.s. Let $r > 0$.

$$P[\sup_{j \geq k_n} \|S_j - S_{k_n}\| \geq r] = \lim_{\ell \to \infty} \underbrace{P[\max_{k_n \leq j \leq \ell} \|S_j - S_{k_n}\| \geq r]}_{\leq 2P[\|S_\ell - S_{k_n}\| \geq r]} \xrightarrow[\text{(iv)}]{n \uparrow \infty} 0$$

$$\leq 2(P[\|S_\ell - S\| \geq r/2] + P[\|S_{k_n} - S\| \geq r/2])$$

$$\Rightarrow P\left[\limsup \|S_k - S\| \geq 2r\right] = \lim_{n\to\infty} \underbrace{P\left[\sup_{j\geq k_n} \|S_j - S\| \geq 2r\right]} = 0$$

$$\leq \underbrace{P\left[\sup_{j\geq k_n} \|S_j - S_{k_n}\| \geq r\right]}_{\to 0} + \underbrace{P\left[\sup_{j\geq k_n} \|S_{k_n} - S\| \geq r\right]}_{\to 0 \text{ since } S_{k_n} \to S \text{ a.s.}}$$

<u>L^p convergence</u> : via Lemma 28, Exercise. $\square$

3. Analysis on Gaussian measure spaces

Ref. Ledoux. Conc. of measure and logarithmic Sobolev inequalities, Lecture Notes Berlin 1997

3.1. Isoperimetric inequalities

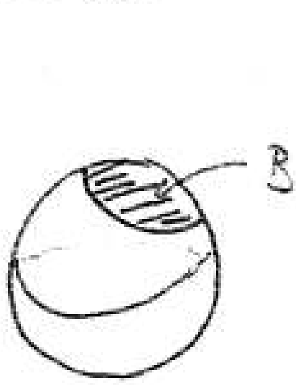
a) Classical isoperimetric inequality on unit sphere $S^d \subseteq \mathbb{R}^{d+1}$:

σ_d = uniform distribution on S^d,

$$\sigma_d^+(A) := \liminf_{r \downarrow 0} \frac{\sigma_d(A_r) - \sigma_d(A)}{r}$$ boundary measure, Minkowski content of $A \in \mathcal{B}(S^d)$

$$A_r := \{x \in S^d : d(x, A) < r\}$$ r-neighbourhood

THEOREM (E. Schmidt 1948, P. Lévy 1951)



$$(*) \quad \sigma_d^+(A) \geq \sigma_d^+(B)$$

for any $A \in \mathcal{B}(S^d)$, $B \subset S^d$ ball with $\sigma_d(B) = \sigma_d(A)$

"Balls minimize surface area for given volume"

Equivalent statement: (**) $\sigma_d^+(A) \geq I(\sigma_d(A))$, "=" for balls,

where $I(v) :=$ surface measure of ball with volume v (w.r.t. σ_d)

I = isoperimetric function of σ_d

Integrated form of isoperimetric inequality:

$$(***) \quad \sigma_d(A_r) \geq \sigma_d(B_r)$$

for any $A \in \mathcal{B}(S^d)$, $B \subset S^d$ ball with $\sigma_d(B) = \sigma_d(A)$, $r \geq 0$.

<u>Exercise</u> Prove that (*) and (***) are equivalent.

Hint: First assume that A is a finite union of balls in S^d.

<u>COROLLARY</u> (Concentration of measure)

$A \in \mathcal{B}(S^d)$ s.t. $\sigma^d(A) \geq 1/2$

$$\Rightarrow \sigma^d(A_r) \geq 1 - \sqrt{\tfrac{\pi}{8}}\, e^{-(d-1)r^2/2} \quad \forall r \geq 0$$

Proof: Estimate measure of cap B_r by induction, cf. Ledoux: The conc. of meas. phen., Thm "measure concentrates in $O(\frac{1}{\sqrt{d}})$ neighbourhood of A"

e.g. A = northern / southern hemisphere

$\leadsto$ measure concentrates in $O(\frac{1}{\sqrt{d}})$ neighbhd. of equator

$$\sigma_d(d(\cdot, \text{equator}) \geq r) \leq \sqrt{\tfrac{\pi}{2}}\, e^{-(d-1)r^2/2}$$

b) Gaussian isoperimetric inequality

$\sigma_d^r :=$ uniform distribution on $r \cdot S^d \subseteq \mathbb{R}^{d+1}$

Poincaré's lemma: For any $n \in \mathbb{N}$,

$$\sigma_d^{\sqrt{d}} \circ (x_1, \dots, x_n)^{-1} \xrightarrow{w} N(0, I_n) =: \gamma^n$$

"Standard normal distribution is finite dimensional projection of uniform distribution on infinite dimensional sphere"

Proof: elementary, cf. e.g. Georgii: Stochastik, Ch. 2.

Apply limit to (*), (**), (***):

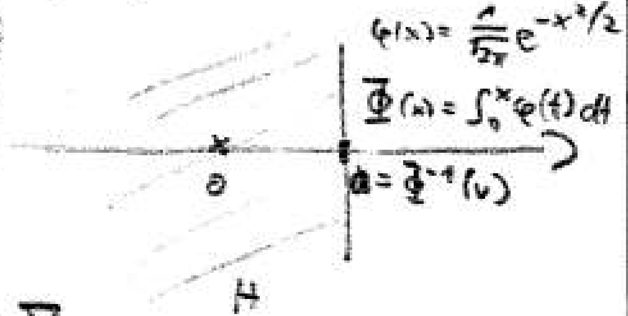
(I1) $\gamma^{n,+}(A) \geq \gamma^{n,+}(H)$

for any $A \in \mathcal{B}(\mathbb{R}^n)$, $H \subset \mathbb{R}^n$ half-space with $\gamma^n(H) = \gamma^n(A)$

$$\boxed{\text{(I2)} \quad \gamma^{n,+}(A) \geq I(\gamma^n(A))} \quad \text{for any } A \in \mathcal{B}(\mathbb{R}^n)$$

where $I(v) :=$ surface measure of half-space with measure v

$$\boxed{I(v) = \varphi\left(\Phi^{-1}(v)\right)}$$



DIMENSION - INDEPENDENT !

$$\boxed{\text{(I3)} \quad \gamma^n(A_r) \geq \gamma^n(H_r) = \Phi(a+r), \quad a = \Phi^{-1}(\gamma^n(A))}$$
$A \in \mathcal{B}(\mathbb{R}^n), r \geq 0$

<u>THEOREM 31</u> (GAUSSIAN ISOPERIMETRIC INEQUALITY)

For any $n \in \mathbb{N}$, $A \in \mathcal{B}(\mathbb{R}^n)$, $r, a \geq 0$:

$$\gamma^n(A) \geq \Phi(a) \implies \gamma^n(A_r) \geq \Phi(a+r)$$

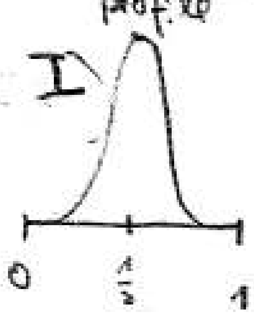
- First proof via isoperimetric ineq. on sphere (Borell, Sudakov–Tsirelson)
- Direct proof by Ehrhard '83
- Proof via CLT by Bobkov '97, cf. Thm. 33 below
- Semigroup proof by Bakry & Ledoux '96, extends to diffusion generators

c) Functional form of Gaussian isoperimetric inequality:

$$\boxed{\text{(I4)} \quad I\left(\int f\,d\gamma^n\right) - \int I(f)\,d\gamma^n \le \int |\nabla f|\,d\gamma^n \quad \text{for any (smooth) Lipschitz function } f: \mathbb{R}^n \to [0,1]}$$

Lemma 32 (Bobkov 1996)

Gaussian isoperimetric profile



The inequalities (I2), (I3) and (I4) are all equivalent.

Proof (I4) $\Rightarrow$ (I2): Assume (w.l.o.g.) A is finite union of open balls,

$$f_r(x) := \left(1 - \tfrac{1}{r}\, d(x,A)\right)^+ \text{ cut-off function}, \quad r > 0.$$

$$f_r = 1 \text{ on } A, \quad f_r = 0 \text{ on } A_r^c, \quad \lim_{r \downarrow 0} f_r = I_A$$

$$\underbrace{I\left(\int f_r\,d\gamma^n\right)}_{\to \gamma^n(A) \text{ by mon. conv.}} - \int \underbrace{I(f_r)}_{\to 0 \text{ a.e., unif. bdd.}}\,d\gamma^n \le \underbrace{\int |\nabla f_r|\,d\gamma_n}_{= \frac{1}{r}(\gamma_n(A_r) - \gamma_n(A))}$$

$$\overset{r\downarrow 0}{\Longrightarrow} \mathcal{I}(\gamma^n(A)) \leq \liminf_{r\downarrow 0} \frac{1}{r}\left(\gamma_n(A_r)-\gamma_n(A)\right) = \gamma_n(A^+)$$

<u>$(I2) \Rightarrow (I3)$</u>: to show:

$$\underbrace{\Phi^{-1}(\gamma^n(A_r))}_{=:\,h(r)} \geq \Phi^{-1}(\gamma^n(A)) + r \quad \forall r \geq 0$$

W.l.o.g. A finite union of balls, ∂A_r piecewise smooth

$$\Rightarrow \frac{d}{dr}\gamma^n(A_r) = \gamma^{n,+}(A_r)$$

$$\Rightarrow h \text{ differentiable}, \quad h'(r) = \frac{\gamma^{n,+}(A_r)}{\varphi(\Phi^{-1}(\gamma^n(A_r)))} \overset{(I2)}{\geq} 1$$

$$\Rightarrow h(r) = h(0) + \int_0^r h'(s)\,ds \geq \Phi^{-1}(\gamma^n(A)) + r$$

<u>$(I3) \Rightarrow (I4)$</u>: cf. Bobkov 96

$\square$

THEOREM 33 (Bobkov '97) Let $I := \varphi \circ \Phi^{-1}$. Then the inequality

$$(*) \quad I\left(\int f\,d\mu\right) \le \int \sqrt{I(f)^2 + |\nabla f|^2}\,d\mu$$

holds for any Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ provided

1) $\mu = \mathrm{Unif}(\{-1,1\})$, $|\nabla f(x)| = \left|\frac{f(1)-f(-1)}{2}\right|$

2) $\mu = \mathrm{Unif}(\{-1,1\}^n)$, $|\nabla f(x)|^2 = \sum_{i=1}^{n} \left|\frac{f(x_1,\dots,x_{i-1},-x_i,x_{i+1},\dots,x_n)-f(x)}{2}\right|^2$

3) $\mu = \gamma^n$, ∇f = Euclidean gradient.

In particular, (I_4) holds w.r.t. μ.

Rem 1) $(*)$ is in general stronger than (I_4)

2) Dimension independent $\rightsquigarrow$ carries over to Gaussian measures on Banach sp

3) $\exists$ extension to strictly log concave measures (Bakry/Ledoux)

Proof 1) $f : \{-1,1\} \to \mathbb{R}$, $a = f(1)$, $b = f(-1)$

$$(*) \Leftrightarrow I\left(\frac{a+b}{2}\right) \le \frac{1}{2}\sqrt{I(a)^2 + \left(\frac{a-b}{2}\right)^2} + \frac{1}{2}\sqrt{I(b)^2 + \left(\frac{a-b}{2}\right)^2} \quad (**)\ \forall a,b$$

holds by explicit computation using $(I')^2$ convex, $I \cdot I'' = -1$, cf. Bobkov '97

2) $\mathrm{Unif}(\{-1,+1\}^n) = \nu^n$, $\nu = \mathrm{Unif}(\{-1,+1\})$

Claim: (*) extends from ν to ν^n (Factorization property)

By induction: Suppose (*) holds for $n \in \mathbb{N}$, $f: \{-1,+1\}^{n+1} \to [0,1]$.

$f_-, f_+ : \{-1,+1\}^n \to [0,1]$, $f_\pm(x) := (x, \pm 1)$

$$|\nabla f(x,\pm 1)|^2 = |\nabla f_\pm(x)|^2 + \frac{1}{4}|f_+(x) - f_-(x)|^2$$

$$\Rightarrow \int \sqrt{I(f)^2 + |\nabla f|^2}\, d\nu^{n+1}$$

$$= \frac{1}{2}\int \sqrt{\underbrace{I(f_-)^2 + |\nabla f_-|^2}_{=: A_-^2} + \underbrace{\frac{1}{4}|f_+ - f_-|^2}_{=: B^2}}\, d\nu^n + \frac{1}{2}\int \sqrt{\underbrace{I(f_+)^2 + |\nabla f_+|^2}_{=: A_+^2} + \underbrace{\frac{1}{4}|f_+ - f_-|^2}_{= B^2}}\, d\nu^n$$

$$\overset{(1)}{\geq} \frac{1}{2}\sqrt{\Big(\underbrace{\int A_- d\nu^n}_{\geq I(\int f_- d\nu^n) \text{ by induction hyp.}}\Big)^2 + \Big(\int B\, d\nu_n\Big)^2} + \frac{1}{2}\sqrt{\Big(\underbrace{\int A_+ d\nu^n}_{\geq I(\int f_+ d\nu^n)}\Big)^2 + \Big(\int B\, d\nu^n\Big)^2}$$

$$\overset{(**)}{\geq} I\left(\frac{\int f_- d\nu^n + \int f_+ d\nu^n}{2}\right) = I\left(\int f\, d\nu^{n+1}\right)$$

Proof of (Δ): $\underbrace{\int\sqrt{A^2+B^2}}\geq\underbrace{\sqrt{(\int A)^2+(\int B)^2}}$

$$=\Big(\iint\underbrace{\sqrt{A(x)^2+B(x)^2}\sqrt{A(y)^2+B(y)^2}}\Big)^{1/2}=\Big(\iint\underbrace{A(x)A(y)+B(x)B(y)}\Big)^{1/2}$$

$\geq$

3) $f:\mathbb{R}^n\to[0,1]$, $f\in C_b^2$

CLT: $$\int f\,d\gamma^n=\lim_{k\to\infty}\int\underbrace{f\left(\frac{x_1+\dots+x_k}{\sqrt{k}}\right)}_{=:f_k(x_1,\dots x_k)}\nu^{nk}(dx)$$

Assume $n=1$ for simplicity ($n>1$ analogous). By 2):

$$(***)\quad I\left(\int f_k\,d\nu^k\right)\leq\int\sqrt{\underbrace{I(f_k(x))}_{=I\left(f\left(\frac{x_1+\dots+x_k}{\sqrt{k}}\right)\right)}+\underbrace{|\nabla^{discr.}f_k(x)|^2}_{\overset{(a)}{=}f'\left(\frac{x_1+\dots+x_k}{\sqrt{k}}\right)^2+O(k^{-1/2})}}\;\nu^k(dx)$$

uniformly over $x\in\{-1,1\}^k$

$$\overset{k\to\infty}{\underset{\text{CLT}}{\Longrightarrow}}\quad I\left(\int f\,d\gamma\right)\leq\int\sqrt{I\circ f+|f'|^2}\,d\gamma\quad\forall f\in C_b^2\overset{Ex.}{\Longrightarrow}\text{Claim}$$

I contin. & bdd.

Proof of $(\circ)$:

$$\left|\nabla_i^{\mathrm{discr.}} f_k(x)\right| = \frac{1}{2}\left| f\left(\frac{x_1+\dots+x_k-2x_i}{\sqrt{k}}\right) - f\left(\frac{x_1+\dots+x_k}{\sqrt{k}}\right)\right|$$

$$\overset{f\in C_b^2}{=} \frac{1}{\sqrt{k}} f'\left(\frac{x_1+\dots+x_k}{\sqrt{k}}\right) + \underbrace{O\left(\frac{1}{k}\right)}_{\text{uniform in } x}$$

$$\Rightarrow \left|\nabla^{\mathrm{discr.}} f_k(x)\right|^2 = \sum_{i=1}^{k} \left|\nabla_i^{\mathrm{discr}} f_k(x)\right|^2 = \frac{k}{k} f'\left(\frac{x_1+\dots+x_k}{\sqrt{k}}\right)^2 + \sum_{i=1}^{k} O(k^{-3/2})$$

$\square$

3.2. Isoperimetric inequality and concentration for general Gaussian measures

E sep. Banach space, μ cent. Gaussian measure on $\mathcal{B}(E)$, $H = H(\mu)$

$B := \{ h \in H : \|h\|_H \leq 1 \}$ unit ball

THEOREM 34 Let $a \in \mathbb{R}$, $A \in \mathcal{B}(E)$. Then:

$$\mu(A) \geq \Phi(a) \Longrightarrow \mu(A + rB) \geq \Phi(a+r) \quad \forall r \geq 0$$

where $A + rB := \{ x + rh : x \in A, h \in B \}$.

Corollary 35 $\mu(A) > 0 \Longrightarrow \underbrace{\mu(A+H)}_{= \lim_{r\to\infty} \mu(A+rB)} = 1$

(although $\mu(H) = 0$ if $\dim H = \infty$!)

"H is skeleton of (E, μ)"

In particular: $A = A + H \Rightarrow \mu(A) \in \{0, 1\}$

Proof of Theorem 34: by finite dimensional approximation.

Suppose $Y \sim \mu$, $Y = \sum_{k=1}^{\infty} Z_k e_k$ with Z_k iid $\sim N(0,1)$,

$\{e_k : k \in \mathbb{N}\}$ ONB of $H \subseteq E$

$$(*)\quad Y_n := \sum_{k=1}^{n} Z_k e_k \longrightarrow Y \text{ in } L^p(\Omega \to E; P)$$

Assume for simplicity $A \subseteq E$ closed, f. $\varepsilon > 0$ fixed, $\delta > 0$.

(1) <u>Tightness:</u> $\mu(A) \geq \Phi(a)$

$$\Longrightarrow \exists\, K \subseteq A \text{ compact} : P[Y \in K] = \mu(K) \geq \Phi(a-\varepsilon)$$

Let $K_\delta := \{x \in E : d(x,K) < \delta\}$ (w.r.t. $\|\cdot\|_E$)

(2) <u>Approximation:</u> $\exists\, n_0 \ \forall n \geq n_0$:

$$(**)\quad P[Y_n \in K_\delta] \geq \Phi(a - 2\varepsilon) \qquad (b, (*), (1))$$

$$(***)\quad P[Y \in K_{2\delta} + rB] \geq P[Y_n \in K_\delta + rB] - \varepsilon$$
$$\geq P[Y_n \in K_\delta + rB^n] - \varepsilon$$

where $B^n :=$ unit ball in $\underbrace{H^n := H(P \circ Y_n^{-1})}_{\text{Cameron Martin space of } Y_n}$

③ Application of isoperimetric inequality w.r.t. γ^n:

$$S_n : \mathbb{R}^n \to H^n \text{ isometry}, \quad S_n(z_1,\dots,z_n) = \sum_{k=1}^n z_k e_k$$

in particular $S_n(\underbrace{B(0,r)}_{\text{Euclidean ball}}) = rB^n$

$$\mu^n := P \circ Y_n^{-1} = \gamma^n \circ S_n^{-1} \quad , n \geq n_0$$

$$\gamma^n(S_n^{-1}(K_\delta)) = \mu^n(K_\delta) = P[Y_n \in K_\delta] \overset{(**)}{\geq} \Phi(a-2\varepsilon)$$

$$\underset{\text{isoper. inequ.}}{\Longrightarrow} \quad \Phi(a-2\varepsilon+r) \leq \gamma^n(S_n^{-1}(K_\delta)_r) = \gamma^n(S_n^{-1}(K_\delta) + B(0,r))$$

$$\leq \mu^n(K_\delta + rB^n) = P[Y_n \in K_\delta + rB^n]$$

$$\overset{(***)}{\leq} P[Y \in K_{2\delta} + rB] + \varepsilon$$

④ $\underline{\delta \downarrow 0}$: K compact

$$\Rightarrow P[Y \in A + rB] \geq P[Y \in K + rB] \geq \Phi(a-2\varepsilon+r) - \varepsilon \quad \forall \varepsilon > 0$$

$\square$

<u>Example</u> $E=\mathbb{R}^d$, $C=\sigma\sigma^T$ covariance matrix $\Rightarrow \mu = \gamma^n \circ \sigma^{-1}$, $rB = \sigma(B(0,r))$ ellipsoid

$$\mu(A) \geq \Phi(a) \Rightarrow \mu(A + \text{ellipsoid}) \geq \Phi(a+r)$$

<u>Application to concentration of measure</u>

$$\alpha(r) := \sup \{ \mu[(A+rB)^c] : A \in \mathcal{B}(E) \text{ s.t. } \mu(A) \geq \tfrac{1}{2} \}$$

concentration function

<u>Corollary 36</u> $\alpha(r) \leq e^{-r^2/2} \quad \forall r \geq 0$ "Gaussian concentration"

<u>Proof</u>: $\mu(A) \geq \frac{1}{2} = \Phi(0)$

$$\Rightarrow \mu(A+rB) \geq \Phi(r) \geq 1 - e^{-r^2/2}. \quad \square$$

<u>Rem.</u>: Gaussian concentration is weaker than Gaussian isoper. ineq.
There are alternative proofs (direct, semigroup, log Sobolev), cf. Ledoux: The conc. of measure phenomenon.

<u>Corollary 37</u> $F: E \to \mathbb{R}$ measurable & <u>H-Lipschitz</u>, $i.e.$, [illegible]

$$\exists\, L \in (0,\infty): |F(x+h) - F(x)| \leq L \cdot \|h\|_H \quad \forall h \in H$$

m median of F

$$\Rightarrow \mu(F \geq m + r) \leq e^{-\frac{r^2}{2L^2}} \quad \forall r \geq 0$$

In particular:

$$\mu(|F-m| \geq r) \leq 2 e^{-\frac{r^2}{2L^2}} \quad \forall r \geq 0$$

Rem. $F: E \to \mathbb{R}$ H-Lipschitz

$\Leftrightarrow$ F Malliavin differentiable with $\|D^H F\|_H \leq 1$ a.e.

Proof of Cor. 37 Wlog $L=1$; otherwise consider F/L.

m median of F $\Rightarrow \mu(F \leq m) \geq \frac{1}{2}$

$$\Rightarrow \mu(F \leq m+r) \underset{\substack{\uparrow \\ F \text{ H-Lipschitz with } L=1}}{\geq} \mu(\{F \leq m\} + B_r) \geq 1 - e^{-r^2/2}$$

$$\Rightarrow \mu(F > m+r) \leq e^{-r^2/2}$$

$r \to r-\varepsilon$, $\varepsilon \downarrow 0$ $\rightsquigarrow$ Claim. □

Concentration w.r.t. $\|\cdot\|_E$ $\quad \bar{\sigma} := \sup_{\|\ell\|_{E^*} \le 1} C(\ell,\ell)^{1/2}$

Lemma 38 $\|x\|_E \le \bar{\sigma} \cdot \|x\|_H \quad \forall x \in E$

In particular, $F(x) = \|x\|_E$ is H-Lipschitz with $L = \bar{\sigma}$.

Proof $\|x\|_H \overset{\text{Def.}}{=} \sup_{C(\ell,\ell)=1} \ell(x) \Rightarrow \ell(x) \le C(\ell,\ell)^{1/2} \|x\|_H \le \bar{\sigma} \|\ell\|_{E^*} \|x\|_H$

for any $\ell \in E^*$, $x \in E$

$$\Rightarrow \|x\|_E = \sup_{\|\ell\|_{E^*} \le 1} \ell(x) \le \bar{\sigma} \|x\|_H \quad \forall x \in E$$

$$\Rightarrow |F(x) - F(y)| \overset{\Delta\text{ineq.}}{\le} \|x-y\|_E \le \bar{\sigma} \|x-y\|_E \quad \forall x,y \in E$$

Corollary 39 μ centered Gaussian measure on E, m median of $\|\cdot\|_E$

$$\Rightarrow \quad \mu(\|x\|_E \ge m + r) \le e^{-\frac{1}{2}\left(\frac{r}{\bar{\sigma}}\right)^2} \quad \forall r \ge 0$$

In particular $\int e^{\alpha \|x\|_E^2} \mu(dx) < \infty \quad \forall \alpha < \frac{1}{2\bar{\sigma}^2}$

(Extension of Fernique's Theorem ∇)

EXAMPLE (Deviations of Gaussian Random fields) $D \subset \mathbb{R}^d$ cp.

$X_s : \Omega \to \mathbb{R}$ $(s \in D)$ centered continuous GRF

$\mu :=$ Law of $(X_s)_{s \in D}$ on $E = C(D, \mathbb{R}) \Rightarrow$

$$P\Big[\sup_{s \in D} X_s^x \geq m + t\Big] \leq e^{-\frac{1}{2}\left(\frac{t}{\bar{\sigma}}\right)^2} \quad \forall t \geq 0$$

where $\bar{\sigma} = \sup_{s \in D} C(s,s)^{1/2} = \sup_{s \in D} \sigma(X_s)$.

Proof: $F(x) = \sup x_s$ is H-Lipschitz with $L = \bar{\sigma}$.

3.3 Images of Gaussian measures

Exercise (Transformations of Gaussian measures in $\mathbb{R}^n$)

$m \in \mathbb{R}^n$, $C \in \mathbb{R}^{n\times n}$ symmetric, non-negative, $\sigma \in \mathbb{R}^{n\times n}$, $h \in \mathbb{R}^n$

(i) Show: $X \sim N(m, C) \Rightarrow \sigma X + h \sim N(m+h, \sigma C \sigma^T)$

(ii) $\mu := N(m, C)$, $\nu := \mu \circ T^{-1}$, $T(x) := \sigma x + h$. Then:

$$\nu \ll \mu \iff h \perp \ker C \text{ and } \ker(\sigma C \sigma^T) = \ker(C)$$

(iii) If $\sigma = I_n$ and $h \perp \ker C$ then

$$\frac{d\nu}{d\mu} = e^{(x-m)\cdot C^{-1}h - \frac{1}{2} h \cdot C^{-1} h}$$

where C^{-1} is inverse on $(\ker C)^\perp$

Extension to Gaussian measures on Banach spaces?

μ centered Gaussian measure on sep. Banach space E

$H = H(\mu)$ Cameron-Martin space

$$E^*/\sim \ \subseteq H^* \overset{i}{\cong} H \subseteq E$$

a) <u>Translations</u> $T_h(x) := x+h$, $h \in E$, $\mu_h := \mu \circ T_h^{-1}$

THEOREM 40 (Cameron-Martin)

$\mu_h \ll \mu \iff h \in H$

In this case: $$\frac{d\mu_h}{d\mu} = \exp\left(X_h - \frac{1}{2}\|h\|_H^2\right)$$

<u>Rem.</u> (E, H, μ) Wiener space: $X_h(\omega) = \int h' \, d\omega$ stoch. integral,
Theorem is special case of Girsanov Theorem

<u>Proof</u> '$\Leftarrow$' via computation of Fourier transforms: $l \in E^*$, $g := i(l) \in$

$$h \in H \setminus \{0\} \Rightarrow g = \underbrace{g - \overbrace{\frac{(g,h)_H}{(h,h)_H}}^{=:\alpha} h + \frac{(g,h)_H}{(h,h)_H} h}_{\text{orthogonal in } H}$$

$$\Rightarrow l = X_g = \underbrace{X_{g-\alpha h} + \alpha X_h}_{\text{independent}} \quad \mu\text{-a.s.}$$

Therefore:

$$\int e^{il}\, e^{X_h - \frac{1}{2}\|h\|_H^2}\, d\mu$$

$$= e^{-\frac{1}{2}\|h\|_H^2} \underbrace{\int e^{iX_{g-\alpha h}}\, e^{(1+i\alpha)X_h}\, d\mu}_{\text{factorizes},\ = e^{-\frac{1}{2}\|g-\alpha h\|_H^2} e^{+\frac{1}{2}(1+i\alpha)^2 \|h\|_H^2}}$$

$$= e^{-\frac{1}{2}\|g\|^2 + i \underbrace{(g,h)_H}_{=l(h)}} = \int e^{il}\, d\mu_h \quad \forall l \in E^*$$

Hence $d\mu_h = e^{X_h - \frac{1}{2}\|h\|_H^2}\, d\mu$.

"$\Rightarrow$" Suppose $\mu_h \ll \mu$, $h \in E$

to show: $h \in H$, i.e. $\|h\|_H = \sup\limits_{\substack{l \in E^* \\ \|l\|_{L^2(\mu)} \le 1}} l(h) < \infty$ $(\in H)$

$E^* \to \mathbb{R}$

ok if $l \mapsto l(h)$ is contin. on E^* w.r.t. $L^2(\mu)$ norm

$l_n \in E^*$ s.t. $\|l_n\|_{L^2(\mu)} \to 0 \Rightarrow l_n \xrightarrow{\mu} 0 \overset{\mu_h \ll \mu}{\Longrightarrow} l_n \xrightarrow{\mu_h} 0$

$\overset{\mu_h \text{ Gaussian}}{\Longrightarrow} \|l_n\|_{L^1(\mu_h)} \to 0 \Rightarrow l_n(h) = \int l_n\, d\mu_h \to 0$ ✓

□

Corollary 41 (Support Theorem)

μ Gaussian measure with mean m $\Rightarrow$ $\operatorname{supp}\mu = \overline{m + H(\mu)}$

Proof Wlog $m=0$.

(i) $\operatorname{supp}\mu = \operatorname{supp}\mu + H(\mu)$:

Indeed: $h \in H(\mu)$: $x \in \operatorname{supp}\mu \overset{\text{Cam. Martin}}{\Longleftrightarrow} x + h \in \operatorname{supp}\mu$

(ii) $\operatorname{supp}\mu \subseteq \overline{H(\mu)}$:

$0 \in \operatorname{supp}\mu \Rightarrow \mu\left(\underline{B}(0,\varepsilon)\right) > 0 \quad \forall \varepsilon > 0$

$\overset{\text{isoper. ineq.}}{\Rightarrow} \mu(B(0,\varepsilon) + H(\mu)) = 1 \quad \forall \varepsilon > 0$

$\Rightarrow \operatorname{supp}(\mu) \subset \overline{B(0,\varepsilon) + H(\mu)} \;\; \forall \varepsilon > 0$. □

Corollary 42 (Zero-one law) μ centered Gaussian

$V \subseteq E$ measurable linear subspace $\Rightarrow \mu(V) \in \{0,1\}$

Proof (i) $H(\mu) \not\subseteq V \Rightarrow \mu(V) = 0$.

$h \in H(\mu) \setminus V \overset{\text{Cam. Martin}}{\Rightarrow} V_\varepsilon := V + \varepsilon h \;\; (\varepsilon > 0)$ disjoint sets with $\mu(V_\varepsilon) > 0$

↯ since $\mu(V_\varepsilon) > 0$ (i.e. $\mu(V_\varepsilon) > \frac{1}{n}$ for some n)
can hold only for countably many disjoint sets.

(ii) $H(\mu) \subseteq V$, $\mu(V) > 0 \Rightarrow \mu(V) = \mu(V + H(\mu)) = 1$
by isoperimetric inequality (Cor. 35). $\square$

b) Linear transformations

when Feldman-Hájek criterion

Example (Dilatations) $\mu_c := \mu \circ D_c^{-1}$, $D_c x := cx$

$\dim(H(\mu)) = \infty$, $c \neq \pm 1 \Rightarrow \mu_c \perp \mu$ singular

Proof: $\dim(H(\mu)) = \infty \Rightarrow \dim(\underbrace{G(\mu)}) = \infty$

$= \overline{E^*/\sim} \subseteq L^2(\mu)$

$\Rightarrow \exists\ l_n \in E^*$ s.t. $\{l_n : n \in \mathbb{N}\}$ orthonormal in $L^2(\mu)$

$\frac{1}{n}\sum_{i=1}^{n} l_i^2 \xrightarrow{LLN} 1$ μ-a.s., whereas

$$\frac{1}{n}\sum_{i=1}^{n}(\underbrace{\ell_i\circ D_c}_{=c\ell_i})^2 \to c^2 \quad \mu\text{-a.s.}$$

$$\Rightarrow \quad \frac{1}{n}\sum_{i=1}^{n}\ell_i^2 \to c^2 \quad \mu_c\text{-a.s.} \quad \lrcorner$$

Remark 1) $\mu_c \perp \mu$ for $c \neq \pm 1$ although $H(\mu_c) = H(\mu)$ $\forall c \neq 0$

2) (B_t) Brownian motion $\Rightarrow$ $\mathrm{Law}(cB_t)_{t\in[0,1]} \perp \mathrm{Law}(B_t)_{t\in[0,1]}$ $\forall c \neq \pm 1$

$[cB]_t = c^2 t$ a.s.

General linear transformations: $\tilde{E}$ sep. Banach space

$\sigma : H(\mu) \to \tilde{E}$ bounded lin. operator,

Assumption: $\exists\, \tilde{\mu}$ centered Gaussian measure on $\tilde{E}$ s.t.

$$\mathrm{Cov}_{\tilde{\mu}}(\ell, k) = \underbrace{(\sigma^*\ell, \sigma^*k)_{H(\mu)}}$$

$\hat{=}$ covariance of image measure in finite dimensional case

<u>Lemma 43</u> $H(\tilde{\mu}) = \sigma(H(\mu))$ and

$$(\sigma g, \sigma h)_{H(\tilde{\mu})} = (g, Ph)_{H(\mu)} \quad \forall g, h \in H(\mu)$$

where P is orth projection onto $\underbrace{(\ker \sigma)^{\perp}}_{= \text{range } \sigma^*}$ in $H(\mu)$.

<u>Proof</u> "$\supseteq$" $h \in H(\mu)$

$$\Rightarrow \|\sigma h\|_{H(\tilde{\mu})} \overset{\text{Def.}}{=} \sup_{\substack{\ell \in \tilde{E}^* \\ \operatorname{var}_{\tilde{\mu}}(\ell) \le 1}} \ell(\sigma h) = \sup_{\substack{\ell \in \tilde{E}^* \\ \|\sigma^* \ell\|_{H(\mu)} \le 1}} (\sigma^* \ell)(h) = \|Ph\|_{H(\mu)}$$

$$\Rightarrow \sigma h \in H(\tilde{\mu}), \quad (\sigma g, \sigma h)_{H(\tilde{\mu})} = (Pg, Ph)_{H(\mu)} = (g, Ph)_{H(\mu)}$$

"$\subseteq$" $x \in H(\tilde{\mu}) \Rightarrow \sup_{\|\sigma^* \ell\|_{H(\mu)} \le 1} \ell(x) < \infty \Rightarrow \ell(x) = 0 \ \forall \ell \in \ker \sigma^*$

$$\Rightarrow x \in \underbrace{\sigma(H(\mu))}_{\text{closed}}$$

$\square$

<u>Rem.</u> Z_n iid $\sim N(0,1)$, $\{\tilde{e}_n : n \in \mathbb{N}\}$ ONB of $H(\tilde{\mu})$

$$\Rightarrow S(\omega) = \sum_{n=1}^{\infty} Z_n(\omega) \tilde{e}_n \text{ conv. a.s. in } \tilde{E}, \ S \sim \tilde{\mu}$$

(of by Karhunen-Loève if $Z_n = X_{\tilde{e}_n} \in \mathcal{E}(\tilde{\mu})$; hence always ok)

THEOREM 4.4 (Extension theorem)

There exists a μ-a.s. unique map $\hat{\sigma}: E \to \tilde{E}$ with the following properties:

(i) $\hat{\sigma}$ is an extension of σ

(ii) $\hat{\sigma}$ is measurable

(iii) $\exists$ measurable linear subspace $V \subseteq E$ s.t. $\mu(V) = 1$ and $\hat{\sigma}$ is linear on V.

Moreover, $\tilde{\mu} = \mu \circ \hat{\sigma}^{-1}$.

"measurable linear transformations of Gaussian measures are specified completely by values on CM space"

Proof 1) Uniqueness: $\hat{\sigma}_1, \hat{\sigma}_2$ sat. (i)-(iii) with V_1, V_2,

$V := V_1 \cap V_2$, $\mu(V) = 1$, $\hat{\sigma}_1, \hat{\sigma}_2$ linear on V

Claim: $\ell \circ \hat{\sigma}_1 = \ell \circ \hat{\sigma}_2$ μ-a.s. on V $\forall \ell \in E^*$

Proof of claim: $V^c := \{x \in V : \underbrace{\ell(\hat{\sigma}_1 x - \hat{\sigma}_2 x)}_{=0 \text{ for } x\in H(\mu)} \leq c\}$, $c \in \mathbb{R}$

$$\Rightarrow V^c = V^c + H(\mu) \overset{\text{Cor. 35}}{\Longrightarrow} \underbrace{\mu(V^c)}_{\substack{\to \mu(V)=1 \text{ as } c\to\infty \\ \to 0 \text{ as } c\to -\infty}} \in \{0,1\} \ \forall c \in \mathbb{R}$$

$$\Rightarrow \exists c_0 : \mu(V^c) = \begin{cases} 0 & \text{for } c < c_0 \\ 1 & \text{for } c > c_0 \end{cases}$$

$c_0 = 0$ since $x \sim -x$ w.r.t. μ

$$\Rightarrow \mu(\{x \in V : \ell(\hat{\sigma}_1 x - \hat{\sigma}_2 x) = 0\}) = 1 \quad \lrcorner$$

2) Existence: $\{e_n\}$ ONB of $(\ker \sigma)^\perp \subseteq H(\mu)$

$$\overset{\text{Lem. 43}}{\Longrightarrow} \{\sigma e_n\} \text{ ONB of } H(\tilde{\mu}),$$

$X_{e_n} \in \mathcal{G}(\mu)$ iid $\sim N(0,1)$

We may assume: • $X_{e_n}(\omega) = (e_n, \omega)_{H(\mu)}$ $\forall \omega \in H(\mu)$.

• X_{e_n} linear on $V_n \subseteq E$ s.t. $\mu(V_n) = 1$

(since $X_{e_n} = \lim_{k\to\infty} \ell_k$ μ-a.s., $\ell_k \in E^*$)

$\hat{\sigma}(\omega) := \sum X_{e_n}(\omega)\, \sigma e_n$ conv. μ-a.s. in $\tilde{E}$

by remark above $\mu \circ \hat{\sigma}^{-1} \sim \tilde{\mu}$

<u>Claim</u>: $\hat{\sigma}$ satisfies (i)-(iii) with $V = \bigcap V_n$

(i) $\omega \in H(\mu) \Rightarrow \hat{\sigma}(\omega) = \sum (e_n, \omega)_{H(\mu)} \sigma e_n = \sigma \omega$

(ii) X_{e_n} meas. $\forall n \Rightarrow \hat{\sigma}$ meas.

(iii) $V = \bigcap V_n$ linear subspace, $\mu(V) = 1$,

X_{e_n} linear on V $\forall n \Rightarrow \hat{\sigma}$ linear on V.

$\square$

<u>Remark</u> (Feldman-Hajek Theorem) Suppose $\tilde{E} = E$

3.4 Gaussian Dirichlet form and Ornstein-Uhlenbeck semigroup

μ Gaussian measure on sep. Banach space E

$H = H(\mu) \subseteq E$ Cameron-Martin space, $E^* \hookrightarrow H^* \overset{i}{\cong} H$

$$\mathcal{F}C_b^\infty(E) := \{x \mapsto f(\ell_1(x), \dots, \ell_n(x)) : n \in \mathbb{N}, \ell_1, \dots, \ell_n \in E^*, f \in C_b^\infty(\mathbb{R}^n)\}$$

"smooth cylinder functions based on E^*"

Directional derivatives: $h \in E$, $F = f(\ell_1, \dots, \ell_n) \in \mathcal{F}C_b^\infty(E)$

$$(\partial_h F)(x) = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(\ell_1(x), \dots, \ell_n(x)) \underbrace{\ell_j(h)}_{= (i(\ell_j), h)_H \text{ if } h \in H} \in \mathcal{F}C_b^\infty(E)$$

Malliavin gradient:

$$(D^H F)(x) := \sum_{j=1}^n \frac{\partial f}{\partial x_j}(\ell_1(x), \dots \ell_n(x)) \, i(\ell_j)$$

$$(h, D^H F)_H = \partial_h F \qquad \forall h \in H$$

Lemma 45 (Integration by parts) For any $F, G \in \mathcal{F}C_b^\infty(E)$ and $h \in H$

$$\int \partial_h F \, G \, d\mu = -\int F \, \partial_h G \, d\mu + \int F G \, X_h \, d\mu$$

Proof enough for $G = 1$ (by product rule)

$$\int \frac{\overbrace{F(x+\varepsilon h)}^{=T_{\varepsilon h}(x)} - F(x)}{\varepsilon} \mu(dx) = \frac{1}{\varepsilon}\left(\int F \, d(\mu \circ T_{\varepsilon h}^{-1}) - \int F d\mu\right)$$

$$\overset{\text{Cam. Martin}}{=} \int \underbrace{F}_{\text{bdd.}} \underbrace{\left(e^{\overbrace{X_{\varepsilon h}}^{\varepsilon X_h \text{ a.s.}} - \frac{\varepsilon^2}{2}\|h\|_H^2} - 1\right)/\varepsilon}_{\overset{\varepsilon \to 0}{\longrightarrow} X_h \text{ a.s., in } L^1(\mu) \text{ along subseqn.}} \, d\mu$$

$$\overset{\varepsilon \downarrow 0}{\Longrightarrow} \int \partial_h F \, d\mu = \int F X_h \, d\mu \qquad \square$$

Remark $h \in H$ is essential

Example $\int \nabla f \, d\gamma^d = \int x \cdot f \, d\gamma^d$ $\forall f \in C_b^\infty(\mathbb{R}^d)$

Dirichlet form on (E, H, μ):

$$\mathcal{E}(F, G) = \frac{1}{2} \int (D^H F, D^H G)_H \, d\mu \,, \quad F, G \in \mathcal{F}C_b^\infty(E)$$

$$\mathcal{E}(F,G) = \frac{1}{2}\sum_k \int \partial_{e_k}F\,\partial_{e_k}G\;d\mu$$

where $\{e_k\}$ is ONB of H.

$\mathbb{D}^{1,2}$:= completion of $\mathcal{F}C_b^\infty$ w.r.t. norm induced by inner prod.

$$(F,G)_{1,2} := \int FG\,d\mu + \mathcal{E}(F,G),$$

$\mathbb{D}^{1,2} \subseteq L^2(E,\mu)$

<u>Theorem 46</u> The quadratic form $(\mathcal{E}, \mathcal{F}C_b^\infty(E))$ is closable (i.e. $\exists!$ continuous extension of $\mathcal{E}$ to $\mathbb{D}^{1,2}$). The self-adjoint linear operator $(\mathcal{L}, D(\mathcal{L}))$ associated to the closed form $(\mathcal{E}, \mathbb{D}^{1,2})$ satisfies $\mathcal{F}C_b^\infty(E) \subseteq D(\mathcal{L})$ and

$$\mathcal{L}F = \frac{1}{2}\sum\left(\partial_{e_k}^2 F - X_{e_k}\partial_{e_k}F\right) \quad \forall F \in \mathcal{F}C_b^\infty(E)$$

<u>Example</u> $E = \mathbb{R}^n$, $\mu = \gamma^n \Longrightarrow C_b^\infty(\mathbb{R}^n) \subseteq D(\mathcal{L})$

$$(\mathcal{L}F)(x) = \frac{1}{2}\Delta F(x) - \frac{1}{2}x\cdot\nabla F(x)$$

(since $X_{e_k}(x) = e_k \cdot x$, e_k ONB of $(\mathbb{R}^n, \text{eucl.})$)

Rem. $F \in \mathcal{F}C_b^\infty(E) \Rightarrow$ w.l.o.g. $\exists n: \partial_{e_k} F = 0 \ \forall k \geq n \Rightarrow \exists\, \mathcal{L}F \in \bigcap_p L^p(\mu)$

Proof. $F, G \in \mathcal{F}C_b^\infty(E)$ 3.31

$$\Rightarrow \mathcal{E}(F,G) = \frac{1}{2} \sum_k \int \partial_{e_k} F \, \overbrace{\partial_{e_k} G}^{\in \mathcal{F}C_b^\infty(E)} \, d\mu$$

$$\overset{\text{ibp}}{=} \frac{1}{2} \sum_k \int F \left(-\partial_{e_k}^2 G + X_{e_k} \partial_{e_k} G \right) d\mu$$

$$\overset{\substack{\text{w.l.o.g.} \\ \text{finite sum}}}{=} - \int F \, \mathcal{L}G \, d\mu$$

In particular: $F_n \to 0$ in $L^2(\mu)$, $\mathcal{E}(F_n - F_m, F_n - F_m) \to 0$

$$\Rightarrow \mathcal{E}(F_n, F_n) \to 0$$

Hence $\mathcal{E}$ extends continuously to $\mathbb{D}^{1,2}$ and

$$\mathcal{E}(F,G) = - \int F \, \mathcal{L}G \, d\mu \qquad \forall F \in \mathbb{D}^{1,2}, \ G \in \mathcal{F}C_b^\infty(E)$$

Semigroup $e^{t\mathcal{L}}$?

Example $E = \mathbb{R}^n$, $\mu = \gamma^n$

$$\mathcal{L} = \frac{1}{2}\Delta - \frac{1}{2} x \cdot \nabla \quad \overset{\text{mart. problem}}{\longleftrightarrow} \quad \text{Ornstein-Uhlenbeck process} \quad dX_t = -\frac{1}{2} X_t \, dt + dB_t$$

Explicit solution by variation of constants (Exercise):

$$X_t = e^{-\frac{t}{2}} X_0 + \int_0^t e^{-\frac{s-t}{2}} \, dB_s$$

Transition semigroup:

$$(P_t f)(x) = E_x[f(X_t)] = E\Big[f\big(e^{-\frac{t}{2}}x + \underbrace{\int_0^t e^{\frac{s-t}{2}}\,dB_s}_{\sim N(0,\int_0^t e^{s-t}ds) = N(0,1-e^{-t})}\big)\Big]$$

Def. For a Gaussian measure μ on a sep. Banach space E, the Ornstein-Uhlenbeck semigroup is defined by the Mehler formula

$$P_t F(x) = \int F\left(\sqrt{e^{-t}}\,x + \sqrt{1-e^{-t}}\,y\right)\mu(dy)$$

for any $t \geq 0$ and $F \in \mathcal{L}^p(\mu)$, $p \in [1,\infty]$.

Rem. $P_t F(x) = \int p_t(x,dy) F(y)$ (Markov semigroup)

where $p_t(x,dy) :=$ Distribution of $\sqrt{e^{-t}}\,x + \sqrt{1-e^{-t}}\,z$ under $\mu(dz)$

is Gaussian measure on E for any $t \geq 0$, $x \in E$

Theorem 47

1) For any $t \geq 0$, P_t is a symmetric non-negative lin. op. on $L^2(E,\mu)$

2) For any $p \in [1,\infty)$, $(P_t)_{t\geq 0}$ is a C_0 contraction semigroup on $L^p(E,\mu)$

3) For any $t \geq 0$, and $P_t(\mathcal{F}C_b^\infty(E)) \subseteq \mathcal{F}C_b^\infty(E)$, and

(*) $\partial_h P_t F = e^{-t/2} P_t \partial_h F$ "Commutation relation"

(*) $\partial_h P_t F = e^{-t/2} P_t \partial_h F \quad \forall h \in H$ "Commutation relation"

4) The generator of $(P_t)_{t\geq 0}$ on $L^2(E,\mu)$ is the unique self-adjoint extension of $(\mathcal{L}, \mathcal{F}C_b^\infty(E))$.

<u>Rem.</u> 1) Thus the generator is $(\mathcal{L}, D(\mathcal{L}))$, and $\mathcal{L}$ is essentially self-adjoint on $\mathcal{F}C_b^\infty(E)$, i.e., $\mathcal{F}C_b^\infty(E)$ is dense in $D(\mathcal{L})$ w.r.t. graph norm.

2) Commutation relation for $\mathcal{L}$: $\partial_h \mathcal{L} F = \mathcal{L} \partial_h F - \frac{1}{2} \partial_h F$

In particular: $\mathcal{L}F = \lambda F \Rightarrow \mathcal{L} \partial_h F = (\lambda - \frac{1}{2}) \partial_h F$ "number operator" of quantum theory

<u>Proof</u> 1) $X, Z \sim \mu$ indep., $Y := e^{-t/2} X + (1-e^{-t})^{1/2} Z$

$$\Rightarrow \mu \otimes p_t = \mathrm{Law}(X,Y) \overset{\text{Exercise}}{=} \mathrm{Law}(Y,X)$$

$$\Rightarrow \mu(dx)\, p_t(x,dy) = \mu(dy)\, p_t(y,dx)$$

$$\Rightarrow \int F\, P_t G \, d\mu = \int P_t F \, G \, d\mu \quad \forall F, G \geq 0.$$

2) $F \in L^\infty(\mu) \Rightarrow P_t F \in L^\infty(\mu)$, $\|P_t F\|_\infty \leq \|F\|_\infty$

$F \in L^1(\mu)$, $F \geq 0 \Rightarrow \int P_t F \, d\mu = \int 1\, P_t F \, d\mu = \int P_t 1\, F \, d\mu = \int F \, d\mu$

$$\overset{P_t \geq 0}{\Rightarrow} P_t F \in L^1(\mu), \ \|P_t F\|_1 \leq \|F\|_1$$

Interpolation $\Rightarrow \|P_t F\|_p \leq \|F\|_p \quad \forall p \in [1,\infty]$

In general: $|P_t F| \leq P_t|F| \Rightarrow \|P_t F\|_p \leq \| P_t|F|\|_p \leq \|F\|_p$

$\lim_{t\downarrow 0} \|P_t F - F\|_{L^p} = 0 \quad \forall p \in [1,\infty),\ F \in L^p(\mu)$ by Lebesgue

3) $F = f(\ell_1, \dots, \ell_n) \in \mathcal{F}C_b^\infty(E)$

$$\Rightarrow P_t F(x) = \int f\left(\sqrt{e^{-t}}\,\ell_1(x) + \sqrt{1-e^{-t}}\,\ell_1(y), \dots\right) \mu(dy)$$

$$= g(\ell_1, \dots, \ell_n) \quad , g \in C_b^\infty(\mathbb{R}^n)$$

Moreover: $h \in H \Rightarrow$

$$\Rightarrow \partial_h P_t F(x) \overset{\text{Dom. conv.}}{=} \int \sqrt{e^{-t}}\, \partial_h F\left(\sqrt{e^{-t}}\,x + \sqrt{1-e^{-t}}\,y\right) \mu(dy)$$

$$= e^{-t/2}\, P_t \partial_h F(x)$$

4) $\mathcal{F}C_b^\infty(E) \subset L^2(E,\mu)$ dense

$$\overset{\text{semigr. theory}}{\Longrightarrow} \underbrace{P_t(\mathcal{F}C_b^\infty(E))}_{\subseteq \mathcal{F}C_b^\infty(E) \text{ by } 3)} \subset \mathrm{Dom}(\text{Generator}) \text{ dense w.r.t. graph norm}$$

$\Rightarrow$ Generator essentially self-adjoint on $\mathcal{F}C_b^\infty(E)$

(i.e. there is only one self-adjoint extension)

Identification of generator on $\mathcal{F}C_b^\infty(E)$:

$F = f(l_1, \ldots, l_n)$, w.l.o.g. $(l_i, l_j)_{L^2(\mu)} = \delta_{ij}$, $l := (l_1, \ldots, l_n)$

$$\overset{t>0}{\Longrightarrow} \frac{d}{dt} P_t F(x) = \frac{d}{dt} \underbrace{\int F(e^{-t/2}x + (1-e^{-t})^{1/2} y)\, \mu(dy)}$$

$$\overset{l \circ \mu = \gamma^n}{=} \int f(e^{-t/2} l(x) + (1-e^{-t})^{1/2} z)\, \gamma^n(dz)$$

$$\overset{\text{Dom. Conv.}}{=} \int \nabla f\left(e^{-t/2} l(x) + (1-e^{-t})^{1/2} z\right) \cdot \left(-\frac{1}{2} e^{-t/2} l(x) + \frac{e^{-t}}{2(1-e^{-t})^{1/2}} z\right) \gamma^n(dz)$$

$$\overset{\text{i.b.p.}}{=} \frac{1}{2} \int \left(-e^{-t/2} l(x) \cdot \nabla f(\ldots) + e^{-t} \Delta f(\ldots)\right) \gamma^n(dz) \text{ in } L^2$$

$$\boxed{\begin{aligned} &\int z \cdot \nabla g(z)\, \gamma^n(dz) \\ &= \int \Delta g(z)\, \gamma^n(dz) \end{aligned}}$$

$$\xrightarrow[\text{z.l.s.}]{t \downarrow 0} \frac{1}{2}\left(-l(x) \cdot \nabla f(l(x)) + \Delta f(l(x))\right) = (\mathcal{L}F)(x)$$

$$\Rightarrow \frac{P_\varepsilon F - F}{\varepsilon} = \frac{1}{\varepsilon} \int_0^\varepsilon \frac{d}{dt} P_t F\, dt \rightarrow \mathcal{L}F \text{ as } \varepsilon \downarrow 0$$

$\square$

3.5. Semigroup approach to functional inequalities

Ledoux: Conc. of measure and log. Sobolev inequalities

Royer: An initiation to log. Sobolev inequalities

Setup in Gaussian case:

μ Gaussian measure on sep. Banach space E, $H = H(\mu)$

$$(P_t F)(x) = \int F(\sqrt{e^{-t}}\, x + \sqrt{1-e^{-t}}\, y)\, \mu(dy) \quad \text{OU-semigroup}$$

Symmetric C_0 contraction semigroup on $L^2(E,\mu)$.

$P_t = e^{t\mathcal{L}}$ where $\mathcal{L}$ is self-adj. operator with quadr. form

$$\mathcal{E}(F,G) = \frac{1}{2}\int_E (D^H F, D^H G)_H \, d\mu \quad , \quad F, G \in \mathbb{D}^{1,2}$$

$\mathbb{D}^{1,2} :=$ completion of $\mathcal{F}C_b^\infty(E)$ w.r.t. $(1,2)$ norm,

$D^H F$ gradient w.r.t. $(\cdot,\cdot)_H$, $(h, D^H F)_H = \partial_h F$

Commutation relations: $P_t(\mathcal{F}C_b^\infty) \subseteq \mathcal{F}C_b^\infty$,

$$(*) \quad \partial_h P_t F = e^{-t/2} P_t \partial_h F \quad \forall h \in H,\ t \geq 0,\ F \in \mathbb{D}^{1,2}$$

$$(**) \quad \partial_h \mathcal{L} F = \mathcal{L}\partial_h F - \frac{1}{2}\partial_h F$$

Remark: Extension to non-Gaussian case

e.g. $E = \mathbb{R}^n$, $\mathcal{E}(f,g) = \frac{1}{2}\int \nabla f \cdot \nabla g \; e^{-U} dx$,

$U : \mathbb{R}^n \to [0,\infty)$ sufficiently smooth, strictly convex (outside ball)

$\mathcal{D}(\mathcal{E}) = H^{1,2}(e^{-U}dx) =$ compl. of $C_b^\infty(\mathbb{R}^n)$ w.r.t. weighted (1,2) norm

Generator: $\mathcal{E}(f,g) = -\int f \mathcal{L} g \; e^{-U} dx$,

$$\mathcal{L} g = \frac{1}{2}\Delta g - \frac{1}{2}\nabla U \cdot \nabla g \quad \text{for } g \in C_b^\infty(\mathbb{R}^n)$$

$(\mathcal{L}, C_0^\infty(\mathbb{R}^n))$ is essentially self-adjoint, i.e.,

$C_b^\infty(\mathbb{R}^n) \subseteq \mathcal{D}(\mathcal{L})$ dense w.r.t. graph norm.

Semigroup: $P_t = e^{t\mathcal{L}} \longleftrightarrow dX_t = dB_t - \frac{1}{2}\nabla U(X_t)\,dt$

Langevin diffusion

Commutation relations:

$$\nabla \mathcal{L} f = \mathcal{L} \nabla f - \frac{1}{2}\nabla^2 U \cdot \nabla f$$

(e.g. $\kappa = 1$ for OU)

$$\nabla f \cdot (\nabla \mathcal{L} f - \mathcal{L}\nabla f) \le -\frac{\kappa}{2}|\nabla f|^2 \quad \text{if } \nabla^2 U \ge \kappa \cdot I$$

$$\Rightarrow \quad |\nabla P_t f| \leq e^{-\kappa t/2} P_t |\nabla f|$$

KEY IDEA:

Short time behaviour : $P_t F \to F$ in $L^2(\mu)$ as $t \downarrow 0$

Long time behaviour : $P_t F \to \int F \, d\mu$ in $L^2(\mu)$ as $t \uparrow \infty$

Relate by considering $\frac{d}{dt} P_t F$

$\rightsquigarrow$ Functional inequalities

LEMMA 48 $I \subseteq \mathbb{R}$ interval, $u \in C_b^2(I)$, $F_t := P_t F$ with $F : E \to I$, $F \in \mathcal{L}^2(\mu)$. Then:

$$\exists \; \frac{d}{dt} \int u(F_t) \, d\mu = -\frac{1}{2} \int u''(F_t) \, \overset{\overset{(*)}{\leq} e^{-t}(P_t \|D^H F\|_H)}{\overbrace{\|D^H F_t\|_H^2}} \, d\mu \quad \forall t > 0$$

Example: $\frac{d}{dt} \int |P_t F|^2 \, d\mu = - \int \|D^H P_t F\|_H^2 \, d\mu$

Proof: $F_t = \overset{\text{Spectral thm.}}{e^{t\mathcal{L}}} F \in D(\mathcal{L})$, $\frac{d}{dt} F_t = + \mathcal{L} F_t$ in $L^2(\mu)$

$$\| u(\underbrace{F_{t+\varepsilon}) - u(F_t) - u'(F_t)}_{\text{bounded}} \underbrace{(F_{t+\varepsilon} - F_t)}_{= +\varepsilon \mathcal{L} F_t + o(\varepsilon)} \|_{L^2} \overset{u \in C_b^2}{\leq} \text{const.} \underbrace{\| F_{t+\varepsilon} - F_t \|_{L^2}^2}_{= \| P_\varepsilon F_t - F_t \|_{L^2}^2 = O(\varepsilon^2)}$$

$F_t \in D(\mathcal{L})$

$$\Rightarrow \frac{d}{dt} u(F_t) = \lim_{\varepsilon \to 0} \frac{u(F_{t+\varepsilon}) - u(F_t)}{\varepsilon} = u'(F_t) \mathcal{L} F_t \quad \text{in } \mathcal{L}^2(\mu)$$

$$\Rightarrow \frac{d}{dt} \int u(F_t)\, d\mu = \int \overset{\in \mathbb{D}^{1,2}}{u'(F_t)} \mathcal{L} F_t \, d\mu = -\mathcal{E}(u'(F_t), F_t)$$

Lipschitz $\quad D(\mathcal{L}) \subseteq \mathbb{D}^{1,2}$

$$= -\int (\underbrace{D^H u'(F_t)}_{\underset{\text{chain rule}}{=} u''(F_t) D^H F_t}, D^H F_t)_H \, d\mu$$

$\square$

a) Poincaré inequality / Spectral gap $\alpha(x) = x^2$

THEOREM 49 $\mathrm{Var}_\mu(F) \le 2\,\mathcal{E}(F,F) \quad \forall F \in \mathbb{D}^{1,2}$

Proof
$$\frac{d}{dt}\int F_t^2\,d\mu = 2\int F_t\,\mathcal{L}F_t\,d\mu = -2\int \underbrace{\|D^H F_t\|_H^2}_{\overset{(*)}{=} e^{-t}\|P_t D^H F\|_H^2} d\mu$$
$$\overset{(*)}{\ge} -2e^{-t}\int \left(P_t\|D^H F\|\right)^2 d\mu \ge -2e^{-t}\,\mathcal{E}(F,F)$$

P_t L^2 contraction

$$\overset{-\int_\varepsilon^t \dots}{\Longrightarrow} \quad \int F_\varepsilon^2\,d\mu - \int F_t^2\,d\mu \le 2\,(e^{-\varepsilon} - e^{-t})\,\mathcal{E}(F,F) \quad \forall\, 0<\varepsilon<t<\infty$$

$$\overset{\varepsilon\downarrow 0,\ t\uparrow\infty}{\Longrightarrow} \quad \int F^2\,d\mu - \left(\int F\,d\mu\right)^2 \le 2\,\mathcal{E}(F,F) \qquad \square$$

Corollary 50 $\mathrm{Spec}(-\mathcal{L}) \subseteq \{0\} \cup [\tfrac{1}{2}, \infty)$,

$$\left\| P_t F - \int F\,d\mu \right\|_{L^2(\mu)} \le e^{-\frac{t}{2}} \left\| F - \int F\,d\mu \right\|_{L^2(\mu)} \quad \forall F \in L^2(\mu)$$

Remark/Exercise Similarly for $\alpha(x) = |x|^p$, $1 < p < \infty$

$\Rightarrow$ L^p-Poincaré inequality

b) Concentration of measure $u(x) = e^{\lambda x}$, $\lambda \in \mathbb{R}$

THEOREM 51 $F \in \mathbb{D}^{1,2}$ s.t. $\|D^H F\|_H \leq 1$ a.s.

$$\Rightarrow \mu\left(F \geq \int F d\mu + r\right) \leq e^{-r^2/2} \quad \forall r \geq 0$$

Remark: mean instead of median!

Proof

1) Suppose F bounded. Wlog $\int F d\mu = 0$. Let $\lambda \in \mathbb{R}_+$.

$\Rightarrow e^{\lambda x}$ C_b^2 on $\mathrm{Range}(F)$

$$\overset{\text{Lem. 48}}{\Rightarrow} \frac{d}{dt} \underbrace{\int e^{\lambda F_t} d\mu}_{=: g(t)} = -\frac{\lambda^2}{2} \int e^{\lambda F_t} \underbrace{\|D^H F_t\|_H^2}_{\leq e^{-t}(P_t \underbrace{\|D^H F\|}_{\leq 1})^2 \leq e^{-t}} d\mu$$

$$\geq -\frac{\lambda^2}{2} e^{-t} \int e^{\lambda F_t} d\mu$$

$$\lim_{t \to \infty} g(t) = e^{\lambda \int F d\mu} = 1$$

$\ulcorner$ Solution of $f' = -\frac{\lambda^2}{2} e^{-t} f$, $f(\infty) = 1$:

$\llcorner$ $\Leftrightarrow (\log f)' = -\frac{\lambda^2}{2} e^{-t} \Leftrightarrow \log f = \frac{\lambda^2}{2} e^{-t} \Leftrightarrow f(t) = e^{\frac{\lambda^2}{2} e^{-t}}$

Gronwall

$\Rightarrow \quad g(t) \leq e^{\frac{\lambda^2}{2} e^{-t}} \quad \forall t \in (0, \infty)$

$t \downarrow 0$

$\Rightarrow \quad \underbrace{\int e^{\lambda F} d\mu}_{\geq e^{\lambda r} \mu(F \geq r)} = g(0) \leq e^{\lambda^2/2} \quad \forall \lambda \in \mathbb{R}_+$

$\Rightarrow \quad \mu(F \geq r) \leq \inf_{\lambda \in \mathbb{R}_+} e^{-\lambda r + \lambda^2/2} = e^{-r^2/2}$

2) General F: $F_c := (F \wedge c) \vee (-c) \in \mathbb{D}^{1,2}$, $\|D^H F_c\|_H \leq 1$

Bound for F_c, $c \to \infty$ $\Rightarrow$ Bound for F. $\blacksquare$

c) Logarithmic Sobolev inequality $u(x) = x \log x$, $u(0) := 0$

$$\mathrm{Ent}_\mu(F) := \int u(F)\,d\mu - u\left(\int F\,d\mu\right) \quad (F \geq 0, \in \mathcal{L}^2(\mu))$$

$$\left(= H(F\mu \,|\, \mu) \text{ if } \int F\,d\mu = 1\right)$$

THEOREM 52

$$\mathrm{Ent}_\mu(F^2) \leq 2\,\mathcal{E}(F,F) \quad \forall F \in \mathbb{D}^{1,2} \qquad (LSI)$$

Remark 1) Constant 2 is optimal ($F = e^{\lambda \ell}$, $\ell \in E^*$)

2) can also be proven via CLT similarly to proof of Gaussian isoper. I, in eqn. above factorization property holds

Proof (Sketch) W.l.o.g. $\int F^2\,d\mu = 1$, $\delta \leq F \leq \delta^{-1}$ for some $\delta > 0$

Then $G := F^2 \in \mathbb{D}^{1,2}$, $D^H G = 2F D^H F$. To show:

$$(LSI) \quad \int G \log G\,d\mu \leq 2 \int \|D^H F\|_H^2\,d\mu = \frac{1}{2}\int \frac{\|D^H G\|_H^2}{G}\,d\mu$$

Let $G_t := P_t G$, $\delta^2 \leq G_t \leq \delta^{-2}$ since P_t Markov semigroup

$$\frac{d}{dt}\int G_t \log G_t\,d\mu \overset{\text{Lem. 48}}{=} -\frac{1}{2}\int \frac{1}{G_t}\|D^H G_t\|_H\,d\mu$$

$u'(x) = 1 + \log x$, $u''(x) = \frac{1}{x}$, $u \in C_b^2([\delta^2, \delta^{-2}])$

integration w.r.t. stoch. kernel

$$\geq -\frac{e^{-t}}{2}\int \frac{1}{P_t G}\underbrace{\left(P_t \|D^H G\|\right)^2}_{\overset{\text{Cauchy-Schwarz}}{\leq}\; P_t G \,\cdot\, P_t\left(\frac{\|D^H G\|^2}{G}\right)} d\mu$$

$$\geq -\frac{e^{-t}}{2}\int P_t\left(\frac{\|D^H G\|^2}{G}\right) d\mu$$

$$\overset{P_t\ L^2\text{contr.}}{\geq} -\frac{e^{-t}}{2}\int \frac{\|D^H G\|^2}{G}\, d\mu$$

$$\Longrightarrow \int G \log G \, d\mu = -\int_0^\infty \frac{d}{dt}\int G_t \log G_t \, d\mu \, dt$$

$$\left(\begin{array}{l} G_t \to \int G\, d\mu = \int F^2 d\mu = 1 \text{ in } L^2(\mu) \\ \Rightarrow \int G_t \log G_t \, d\mu \to 0 \quad \text{as } t \to \infty \end{array} \right)$$

$$\leq \frac{1}{2}\int \frac{\|D^H G\|^2}{G}\, d\mu$$

□

<u>Rem.</u> Proof carries over to strictly log-concave prob. measures $\rightsquigarrow$ Bakry-Emery criterion

<u>Corollary 53</u> (Hypercontractivity of OU semigroup)

Let $t \geq 0$ and $p, q \in (1,\infty)$ s.t. $\frac{q-1}{p-1} = e^{2t}$. Then

$$\|P_t F\|_{L^q(\mu)} \leq \|F\|_{L^p(\mu)} \quad \forall F \in L^p(\mu)$$

<u>Remark</u>: 1) $t > 0 \Rightarrow q > p \Rightarrow L^q$ norm stronger than L^p norm

P_t improves integrability, $q \to \infty$ as $t \to \infty$

2) Dimension-independent bound

3) Hypercontractivity $\Leftrightarrow$ Log. Sobolev inequality (Gross 75)

<u>Proof</u>: Exercise, cf. Gross: LSI and contr. prop. of semigroups, LNM15

Hint: consider $\gamma(t) = \left(\int |P_t F|^{q(t)} d\mu \right)^{1/q(t)}$,

compute $\gamma'(t)$ and apply LSI $\square$

<u>Corollary 54</u> (Herbst) $\mu(F \geq E_\mu(F) + r) \leq e^{-r^2/2}$
if F is 1-Lipschitz

Proof: $\Lambda(\lambda) := \log \int e^{\lambda F} d\mu$, LSI $\Rightarrow \Lambda'(\lambda) \leq 1/2 \Rightarrow \int e^{\lambda F} d\mu \leq$ ... $\square$

4. Linear SPDEs Ref. Hairer Ch. 5

GOAL: $dX_t = LX_t\,dt + \sigma\,dW_t$, $X_0 = x_0$, $(\ast)$

$(X_t)_{t\geq 0}$ stoch. process taking values in sep. Banach space $\mathcal{B}$,

L generator of C_0 ~~semigroup~~ $(S_t)_{t\geq 0}$ on $\mathcal{B}$

$(\Rightarrow X_t = S_t X_0$ solves equation for $\sigma = 0)$,

$(W_t)_{t\geq 0}$ cylindrical Wiener process over Hilbert space H,

$\sigma : H \to \mathcal{B}$ bounded linear operator.

Example: $\mathcal{B} = H = L^2(0,1)$, $\sigma = id_H$

L = self-adj. realiz. of $\frac{d^2}{dx^2}$ with Dirichlet b.c. on $(0,1)$.

$\rightsquigarrow$ stochastic heat equation driven by space-time white noise $\eta = \dot{W}_t$

Formal solution by variation of constants ansatz: $X_t = S_t Y_t$

$$X_t = S_t X_0 + \underbrace{\int_0^t S_{t-s}\,\sigma\, dW_s}_{\text{Definition?}}$$

4.1. Integration w.r.t. Wiener processes

Hairer 3.7, Prévôt / Röckner, da Prato

H seperable Hilbert space

Def. A stoch. process $W_t : \Omega \to E$, $t \geq 0$, on $(\Omega, \mathcal{A}, P)$ is called a "cylindrical Wiener process over H" iff $E \supseteq H$ is a sep. Banach space,

$$W_{s+t} - W_s \perp\!\!\!\perp \sigma(W_r \mid 0 \leq r \leq s) \quad \forall\, 0 \leq s \leq t,$$

and

$$W_{s+t} - W_s \sim \sqrt{t}\, Z$$

where $Z : \Omega \to E$ is standard normal w.r.t. $(\cdot,\cdot)_H$

<u>Exercise</u> $(W_t)_{t\geq 0}$ cyl. Wiener process over H

$\Longleftrightarrow$ $\forall t_0 > 0$: $(W_t)_{0\leq t\leq t_0} : \Omega \to C([0,t_0], E)$

is a Gaussian random variable with

$$\mathrm{Cov}\left(\ell(W_s), \tilde{\ell}(W_t)\right) = (s\wedge t)\cdot\left(\ell, \tilde{\ell}\right)_{H^*}$$

$\forall s,t \geq 0, \ell, \tilde{\ell} \in E^*$.

Cameron-Martin space consists of all $h_t = \int_0^t g_s\, ds$ where $g \in L^2([0,t_0] \to H)$, $(h, \tilde{h})_{CM} = \int_0^{t_0} (h_t', \tilde{h}_t')_H\, dt$

<u>Existence</u>. $E \supseteq H$ sep. Banach space s.t. there exists $(\cdot,\cdot)_H$ standard normal distrib. μ on E,

$\{e_n\}$ ONB of H, $(B_t^n)_{t\geq 0}$ indep. Brownian motions on $(\Omega, \mathcal{O}, P)$

$$\Longrightarrow \quad W_t := \sum_n B_t^n e_n$$

converges a.s. in $C([0,t_0], E)$ $\forall t_0 \in \mathbb{R}_+$,

limit is H-cylindrical Wiener process

Images of cylindrical Wiener processes

K sep. Hilbert space, $\{e_n\}$ ONB of H,

$\sigma : H \to K$ Hilbert-Schmidt operator, i.e.,

$$\|\sigma\|_{L^2(H,K)} = \sum \|\sigma e_n\|_K^2 = \operatorname{tr}(\sigma^*\sigma) = \operatorname{tr}(\sigma\sigma^*) < \infty$$

$$\overset{\text{Thm. 44}}{\Longrightarrow} \quad \hat{\sigma} := \sum_n X_{e_n} \sigma e_n \quad : E \to K$$

is μ-a.s. unique meas. extension of σ s.t.
$\hat{\sigma}$ is linear on $V \subseteq E$ with $\mu(V) = 1$.

$\tilde{\mu} := \mu \circ \hat{\sigma}^{-1}$ is Gaussian measure on K, $H(\tilde{\mu}) = \sigma(H)$.

Cor. 55 Let (W_t) cyl. Wiener process over H (on $(\Omega, \mathcal{A}, P)$), $\sigma : H \to K$ Hilbert-Schmidt. Then:

1) $\sigma W_t := \hat{\sigma} W_t : \Omega \to K$ is Wiener process on K w.r.t. $(\cdot,\cdot)_{H(\tilde{\mu})}$.

2) $E[\|\sigma(W_t - W_s)\|_K^2] = \|\sigma\|_{L^2(H,K)}^2 \cdot (t-s) \quad \forall 0 \le s \le t$

<u>Remark</u> Definition of (σW_t) is a.s. independent of the space E where (W_t) has been realized.

<u>Proof</u> 1) Exercise

2) $$E\left[\|\underbrace{\hat{\sigma}(W_t - W_s)}\|_K^2\right] = \sum_n \|\sigma e_n\|_K^2 (t-s)$$

$$= \sum_n \underbrace{X_{e_n}(W_t - W_s)}_{\text{indep.}, \sim N(0,t-s)} \sigma e_n$$

$\square$

Stochastic integrals w.r.t. (W_t).

$W_t : \Omega \to E$ cylindrical Wiener process over H,

e.g. $\Omega = C([0,\infty), E)$, $W_t(\omega) = \omega(t)$

GOAL: $\int_0^t \Phi_s \, dW_s$, $\Phi_s(\omega) \in \underbrace{L_2(H,K)}_{\substack{:= \text{ all Hilbert-Schmidt} \\ \text{operators } \sigma : H \to K}}$

$\mathcal{F}_t := \sigma(W_s \mid 0 \le s \le t)$.

(i) Elementary integrands

$$\Phi_t(\omega) = \sum_{k=0}^{n-1} \sigma_k(\omega) I_{(t_k, t_{k+1}]}(t),$$

$n \in \mathbb{N}$, $0 \le t_0 < t_1 < \dots < t_n$, $\sigma_k : \Omega \to L_2(H,K)$ $\mathcal{F}_{t_k}$-measurable

Range (σ_k) finite

$$\int_0^t \Phi_s \, dW_s := \underbrace{\sum_{k=0}^{n-1} \sigma_k \big(\overbrace{W_{t_{k+1} \wedge t} - W_{t_k \wedge t}}^{=: \Delta_k}\big)}_{\in K}$$

$$\int_0^{\cdot} \Phi_s \, dW_s : \Omega \to C(\mathbb{R}_+, K)$$

Theorem 56

1) $M_t = \int_0^t \Phi_s \, dW_s$ is a continuous square-integrable K-valued $(\mathcal{F}_t)$ martingale, i.e.,

(i) $E[\|M_t\|_K^2] < \infty \quad \forall t \geq 0$

(ii) M_t is $\mathcal{F}_t$-meas. $\forall t \geq 0$

(iii) $E[M_t \mid \mathcal{F}_s] = M_s$ a.s. $\forall 0 \leq s \leq t$.

2) $$\frac{1}{2} E\Big[\sup_{s \leq t} \|M_s\|_K^2\Big] \underset{\substack{\uparrow \\ \text{MAXIMAL} \\ \text{INEQUALITY}}}{\leq} E[\|M_t\|_K^2] \underset{\substack{\uparrow \\ \text{ITÔ} \\ \text{ISOMETRY}}}{=} E\Big[\int_0^t tr(\Phi_s \Phi_s^*) \, ds\Big]$$

Proof: Itô isometry:

$$E[\|M_t\|_K^2] = \sum_{k,l} E[\underbrace{(\sigma_k \Delta_k, \sigma_l \Delta_l)_K}_{\cancel{= (\Delta W_k, \sigma_k^* \sigma_l}}]$$

$$= \sum_k E[\underbrace{\|\sigma_k \Delta_k\|_K^2}_{\overset{!}{=} \|\sigma_k\|^2_{L^2(H,K)} \Delta t_k}] + \sum_{k \neq l} E[\underbrace{(\sigma_k \Delta_k, \sigma_l \Delta_l)_K}_{\overset{!}{=} 0}]$$

$$= \int_0^t \|\Phi_s\|^2_{L^2(H,K)}\, ds \quad < \infty \;,$$

Since $E[\|\sigma_k \Delta_k\|_K^2 \mid \mathcal{F}_{t_k}](\omega) = E[\|\sigma_k(\omega)\Delta_k\|_K^2]$

meas indep.

$\cancel{E[\ldots \mid \mathcal{F}_{t_k}]}$

$$\overset{\text{Cor. 55}}{=} \|\sigma_k(\omega)\|^2_{L^2(H,K)} \Delta t_k \;,$$

$$E[(\sigma_k\Delta_k, \sigma_\ell\Delta_\ell) \mid \mathcal{F}_{t_\ell}](\omega) = E[(\sigma_k(\omega)\Delta_k(\omega), \sigma_\ell(\omega)\Delta_\ell)_K$$

$$= 0 \qquad \forall k < \ell$$

Martingale property: Exercise

Maximal inequality: (M_t) K-valued martingale

$\Rightarrow$ $\|M_t\|_K$ real-valued submartingale $\Rightarrow$ max. inequ. holds

$(\|x\|_K = \sup_n \ell_n(x)$ with $\ell_n \in K^*$, s.t.

$$\underbrace{E[\|M_t\|_K \mid \mathcal{F}_s]}_{\sup \ell_n(M_t)} \geq \sup_n \underbrace{E[\ell_n(M_t) \mid \mathcal{F}_s]}_{=\ell_n(M_s)} = \|M_s\|$$

$\square$

<u>(ii) General predictable integrands</u> Fix $a \in (0,\infty)$.

$$\mathcal{F}_t := \sigma\left(W_s \mid 0 \le s \le t\right)$$

$$\mathcal{P}_a := \sigma\left((s,t] \times A : 0 \le s < t \le a,\ A \in \mathcal{F}_s \right)$$

σ-field of predictable subsets of $[0,a] \times \Omega$

(generated by left-contin. adapted processes)

$$L^2_W(0,a) := L^2\left([0,a] \times \Omega \to L_2(H,K),\ \mathcal{P}_a,\ \lambda_{(0,a)} \otimes P\right).$$

square-integrable predictable processes $(t,\omega) \mapsto \Phi_t(\omega)$ taking values in $L_2(H,K)$

<u>Lemma 58</u> The subspace $\mathcal{E}$ of elementary predictable processes is dense in $L^2_W(0,a)$.

<u>Proof</u> 1) $\Phi \in L^2_W(0,a) \Rightarrow \exists\, \Phi_n \in \left\{ \sum_{i=1}^k \sigma_i 1_{\Pi_i} : k \in \mathbb{N}, \sigma_i \in L_2(H,K), \Pi_i \in \mathcal{P}_a \right.$

$$\| \Phi_n - \Phi \|_{L^2(\lambda_{(0,a)} \otimes P)} \to 0 \quad \text{as } n \to \infty$$

Therefore it suffices to show:

(*) $\forall\ \Phi = \sigma \cdot I_{\Pi}$ with $\sigma \in L^2(H,K)$, $\Pi \in \mathcal{P}_a$ $\quad \forall \varepsilon > 0$:

$$\exists\ \tilde{\Phi} \in \mathcal{E} \quad : \quad \|\tilde{\Phi} - \Phi\|_{L^2(\lambda_{(0,a)} \otimes P)} < \varepsilon$$

2) $\tilde{\Phi} := \sum_{j=1}^{m} \sigma\, I_{(s_j,t_j] \times A_j} = \sigma\, I_{\bigcup_{j=1}^{m} (s_j,t_j] \times A_j}$

$m \in \mathbb{N}$, $(s_j,t_j] \times A_j$ disjoint, $A_j \in \mathcal{F}_{s_j}$

$$\|\tilde{\Phi} - \Phi\|_{L^2}^2 = \|\sigma\|^2_{L^2(H,K)} \underbrace{(\lambda_{(0,a)} \otimes P)\left(\Pi \,\Delta \bigcup_{j=1}^{m} (s_j,t_j] \times A_j\right)}_{\overset{!}{<}\ \delta = \varepsilon^2 / \|\sigma\|^2} \overset{!}{<} \varepsilon$$

Thus to show:

$\mathcal{D} := \{ \Pi \in \mathcal{P}_a : \forall \delta > 0\ \exists\, m \in \mathbb{N}$, predictable $(s_j,t_j] \times A_j$: $\sqcup < \delta$
$= \mathcal{P}_a$

ok since:

(i) $\mathcal{D}$ is Dynkin system (contains $\emptyset$, complements, countable unions of disjoint sets in $\mathcal{D}$)

(ii) $\Pi_1, \Pi_2 \in \mathcal{D} \Rightarrow \Pi_1 \cap \Pi_2 \in \mathcal{D}$

(iii) $(s,t] \times A \in \mathcal{D}$ $\quad \forall\ 0 \le s < t$, $A \in \mathcal{F}_s$

$$\Rightarrow \mathcal{D} \supseteq \mathcal{D}(\text{pred. rect.}) = \sigma(\text{pred. rect}) = \mathcal{P}_a .$$

$\square$

$M_W^2(0,a) :=$ all (equivalence classes of) continuous square-integrable K-valued $(\mathcal{F}_t)$ martingales $(M_t)_{0\le t\le a}$ on $(\Omega, \mathcal{F}, P)$,

$$\|M\|^2_{M_W^2(0,a)} = E[\|M_a\|_K^2] \overset{(*)}{\geq} \frac{1}{4} E[\sup_{t\le a} \|M_t\|_K^2]$$

$$\Phi \longmapsto M = \int_0^\cdot \Phi_s \, dW_s$$

$$\mathcal{E} \subseteq L_W^2(0,a) \longrightarrow M_W^2(0,a) \quad \text{(Hilbert space by } (*)\text{)}$$

isometry by Theorem 56

<u>Corollary 59</u> (Itô isometry) :

$\exists!$ isometry $\Phi \mapsto \int_0^\cdot \Phi \, dW$ from $L_W^2(0,a)$ to $M_W^2(0,a)$ such that $\int_0^\cdot \Phi_s \, dW_s$ coincides with elementary stochastic integral for $\Phi \in \mathcal{E}$.

<u>Proof</u> 1) $M^2_W(0,a)$ Hilbert space by (*)

2) $\Phi \mapsto \int_0^\cdot \Phi\, dW$ isometry on $\mathcal{E}$ by Thm. 56

3) $\mathcal{E}$ dense in $L^2_W(0,a)$ by Lem. 58

4) $\Rightarrow \exists!$ isometric extension to $L^2_W(0,a)$. $\square$

<u>Rem.</u> A bdd lin. op. $\Rightarrow A \int_0^\cdot \Phi_s\, dW_s = \int_0^\cdot A\Phi_s\, dW_s$.

<u>Exercise (Deterministic integrands)</u>

Let $\Phi : [0,a] \to L^2(H,K)$ square-integrable. Show:

1) $\int_0^\cdot \Phi\, dW$ is K-valued contin. <u>Gaussian process</u> with

$$\mathrm{Var}\left(\left(k, \int_s^t \Phi_r\, dW_r\right)_K \right) = \int_s^t \|\Phi_r^* k\|_H \, dr \quad \forall k \in K$$

$\Phi_r^* : K \to H$ Hilbert space adjoint of $\Phi_r : H \to K$.

2) $$\int_0^t \Phi_r\, dW_r = \sum_{n=1}^\infty \int_0^t \Phi_r e_n \, dW_r^n, \quad \{e_n\} \text{ ONB of } H$$

w.r.t. convergence in $M^2_W([0,a])$ where

$W_r^n := X_{e_n}(W_r)$ are i.i.d. Brownian motions.

4.2. Weak solutions by stochastic convolution

$(B, \|\cdot\|)$ sep. Banach space, H sep. Hilbert space

$\langle \ell, x \rangle := \ell(x)$ for $\ell \in B^*$, $x \in B$ resp. $\ell \in H^*$, $x \in H$.

$(*)$ $dX_t = LX_t \, dt + \sigma \, dW_t$, $X_0 = x_0$ SDE on B

(W_t) cyl. Wiener process over H, $\sigma : H \to B$ bounded lin. op.,

$(L, D(L))$ generator of C_0 semigroup on B, $x_0 \in B$.

Digression to semigroup theory

$$Lx = \lim_{t \downarrow 0} \frac{S_t x - x}{t}, \quad D(L) = \{x \in B : \text{limit exists in } B\}$$

$(L, D(L))$ is densely defined linear operator. Moreover:

Proposition 60 1) $S_t(D(L)) \subseteq D(L)$ and

$$\frac{d}{dt} S_t x = L S_t x = S_t L x \quad \forall x \in D(L), t \geq 0$$

2) $\frac{d}{dt} \langle \ell, S_t x \rangle = \langle L^* \ell, S_t x \rangle \quad \forall \ell \in D(L^*), x \in B, t \geq 0$

4.13

3) $(S_t^*)_{t\geq 0}$ is a C_0 semigroup on $B^+ := \overline{D(L^*)} \subseteq B^*$

with generator

$$L^+ \ell = L^* \ell \ , \quad D(L^+) = \{\ell \in D(L^*) : L^* \ell \in B^+\}.$$

4) B^+ is "weak* - dense" in B^*, i.e.,

$$\forall \ell \in B^* \ \ \exists \ \ell_n \in B^+ : \ \ell_n(x) \to \ell(x) \ \ \forall x \in B$$

Proof: Pazy: Semigroups of linear operators; Hairer: Ch. 4.

Example (P_t) heat semigroup on $B = C(S^1)$, $S^1 = [0,1]/\sim$

$\Rightarrow$ (P_t^*) heat semigroup on finite signed measures on S^1

(P_t) strongly cont. on B, but (P_t^*) not strongly cont. on B^*:

$$\underbrace{P_t^* \delta_x}_{\text{abs. cont.}} \not\to \underbrace{\delta_x}_{\text{singular}} \text{ w.r.t. TV-norm as } t \downarrow 0$$

B^+ = all abs. cont. $\mu \in B$, (P_t^*) is C_0 on B^+.

Weak and mild solutions

Rem. $f \in C([0,\infty), D(L))$, $x_0 \in D(L)$

$$\Rightarrow \quad \frac{dx}{dt} = Lx + f, \quad x(0) = x_0,$$

has unique solution

$$x(t) = S_t x_0 + \int_0^t S_{t-s} f(s)\, ds$$

Formal replacement $f(t)\,dt \to \sigma\, dW_t$:

$$X_t = S_t X_0 + \int_0^t S_{t-s} \sigma \, dW_s \quad \text{solution of } (*)?$$

Def. 1) $X_t : \Omega \to B$ $(t \geq 0)$ is called a <u>weak solution of $(*)$</u> iff for any $t > 0$, $\int_0^t \|X_s\| ds < \infty$ a.s., and

$$(**) \quad \langle l, X_t \rangle = \langle l, X_0 \rangle + \int_0^t \langle L^* l, X_s \rangle ds + \int_0^t \langle \sigma^* l, dW_s \rangle \quad \text{a.s. } \forall l \in D(L^*), t \geq 0.$$

2) X_t is called a <u>mild solution of $(*)$</u> iff

$$(***) \quad \langle l, X_t \rangle = \langle l, S_t X_0 \rangle + \int_0^t \langle l, S_{t-s} \sigma \, dW_s \rangle \quad \text{a.s. } \forall l \in B^*, t \geq 0$$

<u>Remarks</u> 1) $\ell \in B^* \Rightarrow \sigma^*\ell \in H^* \underset{H}{\overset{\text{Riesz}}{\cong}} H$

$$\Rightarrow \langle \sigma^*\ell, \cdot \rangle = \underbrace{(\sigma^*\ell, \cdot)_H}_{\text{rank one operator}} \in L_2(H, \mathbb{R})$$

$$\Rightarrow \int_0^t \langle \sigma^*\ell, dW_s \rangle = \int_0^t (\sigma^*\ell, dW_s)_H \quad \text{well-defined}$$

2) Similarly, $\langle \ell, S_{t-s}\sigma \cdot \rangle \in H^*$ unif. bounded in s

$$\Rightarrow \exists \int_0^t \langle \ell, S_{t-s}\sigma\, dW_s \rangle = \int_0^t (\sigma^* S^*_{t-s}\ell, dW_s)_H$$

3) $\Rightarrow$ $\int_0^t S_{t-s}\sigma\, dW_s$ exists in any Hilbert space $\hat{B} \supseteq B$ s.t.

(o) $$\int_0^t \|S_{t-s}\sigma\|^2_{L_2(H,\hat{B})}\, ds < \infty$$

If $B \subseteq \hat{B}$ and (X_t) is mild solution then for $\ell \in \hat{B}^* \subseteq B^*$:

$$\langle \ell, X_t - S_t X_0 \rangle = \int_0^t \langle \ell, S_{t-s}\sigma\, dW_s \rangle = \langle \ell, \int_0^t S_{t-s}\sigma\, dW_s \rangle \quad \text{a.s.}$$

B separable

$$\Rightarrow \underbrace{\int_0^t S_{t-s}\sigma\, dW_s}_{\text{stochastic convolution of } \sigma W \text{ with semigroup } (S_t)} = X_t - S_t X_0 \in B \quad \text{a.s.}$$

THEOREM 61

Suppose that $\int_0^t \|X_s\|\,ds < \infty$ a.s. for any $t > 0$.
Then the following assertions are equivalent:

(i) (X_t) is a weak solution of $(*)$.

(ii) (X_t) is a mild solution of $(*)$.

(iii) $X_t = S_t x_0 + \int_0^t S_{t-s}\,\sigma\, dW_s$ a.s. $\forall t \geq 0$

where the stochastic convolution is defined in a Hilbert space $\hat{B} \supseteq B$ satisfying $(\circ)$.

Proof W.l.o.g. $x_0 = 0$, otherwise consider $\tilde{X}_t := X_t - S_t x_0$.
For simplicity we assume that L^* is densely defined (i.e., $B^+ = B$, $\mathcal{L}^+ = \mathcal{L}^*$)

(ii) $\Leftrightarrow$ (iii): by Remark 3 above.

(ii) $\Rightarrow$ (i): (X_t) mild solution, $\ell \in D(L^*)$

Then, by $(***)$,

$$\int_0^t \underbrace{\langle L^* \ell, X_s \rangle}\, ds \overset{\text{Stoch. Fubini Theorem, proof omitted}}{=} \int_0^t \langle \underbrace{\int_r^t S^*_{s-r} L^* \ell \, ds}, \sigma dW_r \rangle$$

$$\overset{(***)}{=} \int_0^s \langle L^* \ell, S_{s-r} \sigma dW_r \rangle \qquad = \int_0^{t-r} \underbrace{S^*_u L^* \ell}_{= \frac{d}{du} S^*_u \ell \text{ for } \ell \in D(L^*) \overset{\text{Assumpt.}}{=} D(L^*)} du = S^*_{t-r} \ell - \ell$$

$$= \int_0^t \langle S^*_{t-r} \ell - \ell, \sigma dW_r \rangle = \int_0^t \langle \ell, S_{t-r} \sigma dW_r \rangle - \int_0^t \langle \ell, \sigma dW_r \rangle$$

$$\overset{(***)}{=} \langle \ell, X_t \rangle - \int_0^t \langle \sigma^* \ell, dW_r \rangle \quad \text{a.s. } \forall t \geq 0.$$

Hence $(**)$ holds for any $\ell \in D(L^*)$.

<u>(i) $\Rightarrow$ (ii)</u>. (X_t) weak solution, $t > 0$ fixed

$$\Rightarrow \quad d\langle \ell, X_s \rangle = \langle L^* \ell, X_s \rangle ds + \langle \ell, \sigma dW_s \rangle \text{ a.s. } \forall \ell \in D(L^*)$$

$$\underset{(\circ\circ)}{\overset{\text{cf. below}}{\Longrightarrow}} \quad d\langle f_s, X_s \rangle = \langle \dot{f}_s + L^* f_s, X_s \rangle dt + \langle f_s, \sigma dW_s \rangle \text{ a.s. on } [0,t]$$

for any $f \in C^1([0,t], B^*)$ s.t. $\mathcal{L}^* f \in C([0,t], B^*)$

Now choose $f_s := S^*_{t-s}\ell$ with $\ell \in D(L^*)$ $\left(\overset{\text{Ass}}{=} D(L^+)\right)$

$$\Rightarrow \dot{f}_s = -L^* f_s$$

$$\Rightarrow d\langle f_s, X_s\rangle = \langle f_s, \sigma\, dW_s\rangle \quad \text{a.s. on } [0,t]$$

$$\overset{\int_0^t}{\Rightarrow} \langle \ell, X_t\rangle - \langle \ell, S_t X_0\rangle = \int_0^t \langle \ell, S_{t-s}\sigma\, dW_s\rangle \quad \text{a.s.}$$

$D(L^*)$ dense in B^* $\Rightarrow$ Equation holds for any $\ell \in B^*$

$\Rightarrow (X_t)$ mild solution.

Proof of (oo):

① ok for $f_s \equiv \ell$, $\ell \in D(L^*)$

② also ok for $f_s = \varphi_s \ell$, $\varphi \in C^1([0,t], \mathbb{R})$, $\ell \in D(L^*)$:

$$d\langle f_s, X_s\rangle = d(\underset{\overset{\uparrow}{C^1}}{\varphi_s} \langle \ell, X_s\rangle) \underset{\substack{\text{Itô} \\ \text{formula/} \\ \text{Stieltjes} \\ \text{prod. rule}}}{=} \langle \ell, X_s\rangle d\varphi_s + \varphi_s\, d\langle \ell, X_s\rangle$$

$$\overset{\substack{\text{weak} \\ \text{sol}}}{=} \langle \underbrace{\dot{\varphi}_s \ell + \varphi_s L^*\ell}_{= \dot{f}_s + L^* f_s}, X_s\rangle ds + \langle \underbrace{\varphi_s \ell}_{= f_s}, \sigma\, dW_s\rangle$$

③ also ok for linear combinations of functions in ②, these are dense in $C^1([0,t], B^*) \cap C([0,t], D(L^*))$.

$\square$

COROLLARY 62 (Existence & uniqueness of weak solutions)

1) Any two weak solutions $(X_t), (\tilde{X}_t)$ on a given setup $(\Omega, \mathcal{F}, P, (W_t))$ with $X_0 = \tilde{X}_0$ a.s. are indistinguishable, i.e., $X_t = \tilde{X}_t \ \forall t \geq 0$ a.s.

2) If B is a Hilbert space then a weak solution with initial condition x_0 exists on B provided

$$\underbrace{\int_0^t \| S_r \sigma \|^2_{L_2(H,B)} \, ds}_{= \mathrm{tr}_B \int_0^t S_r \sigma \sigma^* S_r^* \, ds = \mathrm{tr}_H \int_0^t \sigma^* S_r^* S_r \sigma \, ds} < \infty \quad \forall t \geq 0$$

In this case,

$$(****) \quad X_t = S_t X_0 + \int_0^t S_{t-s} \sigma \, dW_s \quad \forall t \geq 0 \text{ a.s.}$$

Proof. 1) (X_t), $(\tilde{X}_t)$ weak sol. $\overset{\text{Thm. 61}}{\Longrightarrow}$ mild sol.

$\overset{X_0=\tilde{X}_0 \text{ a.s.}}{\Longrightarrow}$ $\langle \ell, X_t\rangle = \langle \ell, \tilde{X}_t\rangle$ a.s. $\forall \ell \in B^*, t \geq 0$

$\overset{B \text{ separable}}{\Longrightarrow}$ $X_t = \tilde{X}_t \quad \forall t \geq 0$ a.s.

2) $\int_0^t \|S_{t-s}\sigma\|^2_{L^2(H,B)}\, ds = \int_0^t \|S_r \sigma\|^2_{L^2(H,B)}\, ds < \infty$

$\Longrightarrow \exists \int_0^t S_{t-s}\sigma\, dW_s \in L^2(\Omega \to B, \mathcal{O}, P)$

$\overset{\text{Thm. 61}}{\Longrightarrow}$ $(****)$ is weak solution

3) $\|A\|^2_{L^2(H,B)} = \operatorname{tr}_B(AA^*) = \operatorname{tr}_H(A^*A)$,

$\int_0^t \operatorname{tr}_B(S_r \sigma \sigma^* S_r^*)\, dr = \operatorname{tr}_B \int_0^t S_r \sigma\sigma^* S_r\, dr$

by monotone convergence

$\square$

Remark X_0 deterministic

$\Longrightarrow$ mild solution $X_t = S_t X_0 + \int_0^t S_{t-s}\sigma\, dW_s$ is Gaussian process

EXAMPLE (Stochastic heat equation) $U \subset \mathbb{R}^d$ bdd. domain

L = self-adj. realization of Δ with Dirichlet b.c.
on $B = L^2(U, \lambda^d)$, $D(L) = H^2(U, \lambda^d) \cap H^1_0(U, \lambda^d)$.

$H = L^2(U, \lambda^d)$

$(*)$ $\quad dX_t = LX_t\,dt + (-L)^{-\alpha/2}\,dW_t$, $\quad \dot{W}_t$ space-time white noise

1) $d=1$, $U = (0,\pi)$, $\alpha = 0$:

L has eigenfunctions $e_k(s) = \sqrt{\frac{2}{\pi}}\sin(ks)$, $k \in \mathbb{N}$,

$Le_k = -k^2 e_k$.

$$S_t = e^{tL} \Rightarrow \int_0^t S_r \sigma\sigma^* S_r^* \, dr = \int_0^t e^{2rL} \, dr$$

has eigenvalues $\int_0^t e^{-2rk^2} dr = \frac{1 - e^{-2tk^2}}{2k^2}$

$$\Rightarrow \operatorname{tr} \int_0^t S_r \sigma\sigma^* S_r \, dr = \sum_k \frac{1 - e^{-2tk^2}}{2k^2} < \infty$$

$\Rightarrow \exists!$ weak solution on B

2) $d=2$, $U=(0,\pi)^2$, $\alpha=0$: eigenfunctions $e_k \otimes e_\ell$, $k,\ell \in \mathbb{N}$,

eigenvalues $-(|k|^2+|\ell|^2)$,

$$\operatorname{Tr} \int_0^t S_r \sigma\sigma^* S_r \, dr = \sum_{k,\ell=1}^{\infty} \frac{1}{k^2+\ell^2}\left(1-e^{-2t(k^2+\ell^2)}\right)$$

$$\sim \int_{[1,\infty)^2} \frac{1}{|z|^2}\, dz \sim \int_1^\infty \frac{r}{r^2}\, dr = \infty \quad \forall t>0$$

3) $U=(0,\pi)^d$, $\alpha>0$ (smoothing)

$e_{(k_1,\dots,k_d)} = e_{k_1} \otimes \dots \otimes e_{k_d}$ eigenfunctions of L, eigenvalues $|k|^2$

$$\Rightarrow S_r \sigma\sigma^* S_r^* e_k = |k|^{-2\alpha} e^{-2r|k|^2} e_k$$

$$\Rightarrow \operatorname{tr} \int_0^t S_r \sigma\sigma^* S_r^* = \sum_{k\in\mathbb{N}^d} \frac{1-e^{-2t|k|^2}}{2|k|^{2+2\alpha}} \sim \int_1^\infty \frac{r^{d-1}}{r^{2+2\alpha}}\, dr$$

finite for $\alpha > \frac{d}{2}-1$

In this case $\exists!$ weak solution of (*)

Digression on analytic semigroups and interpolation spaces

$(S_t)_{t\geq 0}$ C_0 semigroup on sep. Banach space B

$(L, D(L))$ generator, closed & densely defined lin. op.

$\rho(L) = \{\lambda \in \mathbb{C} : \lambda - L$ is one to one$\}$ resolvent set

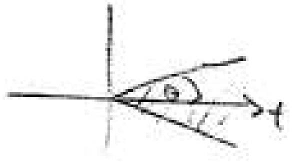
$\sigma(L) = \mathbb{C} \setminus \rho(L)$ spectrum

$G_\lambda = (\lambda - L)^{-1} : B \to D(L) \subseteq B$ bounded lin. op. for $\lambda \in \rho(L)$

Def. (S_t) is called analytic iff the map

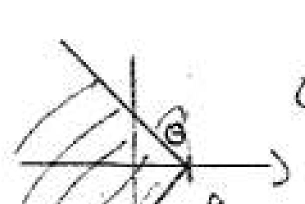
$$t \mapsto S_t, \quad [0,\infty) \to L(B,B) \text{ (bdd. lin. ops.)}$$

has a analytic extension to sector



$\{\lambda \in \mathbb{C} : |\arg \lambda| < \Theta\}$ for some $\Theta > 0$.

FACT 1) (S_t) analytic $\Leftrightarrow$ L satisfies sector condition

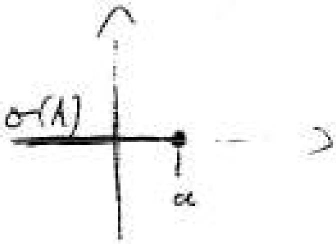


$\sigma(L) \subseteq \{\lambda \in \mathbb{C} : |\arg(a-\lambda)| < \frac{\pi}{2} - \Theta\}$ for some $\Theta \in (0, \frac{\pi}{2})$

and $\|G_\lambda\| \leq \text{const.} \cdot \text{dist}(\lambda, \text{Sector})^{-1}$ $\forall \lambda \in \rho(L)$

2) In this case, $\forall \tilde{\Theta} < \Theta$ $\exists M, a \in \mathbb{R}_+ : \|S_t\| \leq M e^{at}$ whenever $|\arg t| < \tilde{\Theta}$

<u>EXAMPLES</u> 1) H Hilbert space, L self-adjoint,



$L \leq a\,I$ for some $a \in \mathbb{R}$

$\Rightarrow S_t = e^{tL}$ analytic semigroup

2) L generator of analytic semigroup, $\tilde{L} = L + A$

where A satisfies $D(A) \supseteq D(L)$ and

$\forall \varepsilon > 0 \;\exists C > 0 : \|Ax\| \leq \varepsilon \|Lx\| + C\|x\| \;\forall x \in D(L)$

$\Rightarrow \tilde{L}$ is generator of analytic semigroup

e.g. $\Delta + b(x)\cdot\nabla$ with Dirichlet b.c. on $U \subseteq \mathbb{R}^d$

generates analytic semigroup on $L^2(U)$ for $b \in L^d(U) \cup L^\infty(U)$

3) $L = \frac{d}{dx}$, $(S_t f)(x) = f(x+t)$ not analytic on $L^2(\mathbb{R})$, $\operatorname{spec}(L) = i\mathbb{R}$.

Now assume that (S_t) is analytic with $\|S_t\| \leq M e^{at}$ $\forall t > 0$.

<u>Def.</u> $(\lambda - L)^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\lambda t} S_t \, dt \quad (\alpha > 0, \lambda > a)$

<u>is</u> bounded linear operator for $\lambda > a$ since

$$\int_0^\infty t^{\alpha-1} e^{-\lambda t} \|S_t\| \, dt \leq M \int_0^\infty t^{\alpha-1} e^{-(\lambda-a)t} \, dt = M\,\Gamma(\alpha)(\lambda-a)^{-\alpha} < \infty .$$

Rem. L self-adjoint $\Rightarrow$ consistent with definition of $(\lambda-L)^{-\alpha}$ via spectral theorem

$$(\lambda-L)^{\alpha} := \text{inverse of } (\lambda-L)^{-\alpha}$$

in general unbounded linear operator

Def. (Interpolation spaces) $\alpha>0$

$$B_\alpha := \mathrm{Dom}\,(\lambda-L)^{\alpha} = \mathrm{Range}\,(\lambda-L)^{-\alpha} \quad \text{Banach space with}$$

$$\|x\|_\alpha := \|(\lambda-L)^{\alpha}x\|$$

$B_{-\alpha}$:= Completion of B w.r.t.

$$\|x\|_{-\alpha} := \|(\lambda-L)^{-\alpha}x\|$$

$$B_0 := B$$

Fact: The α-norms defined for $\lambda > a$ are all equivalent. In particular, B_α independent of λ (cf. van Neerven)

Example $L = \Delta_{Dir.}$ on $B = L^2(U, dx) \Rightarrow B_\alpha$ Sobolev space $H_0^{2\alpha}$ of order $(2\alpha, 2)$

Remark / Exercise 1) For any $\alpha \geq 0$ and $t \geq 0$, $S_t(B_\alpha) \subseteq B_\alpha$ and 4.26

$$(\lambda - L)^\alpha S_t x = S_t (\lambda - L)^\alpha x \quad \forall x \in B_\alpha$$

2) (S_t) is strongly continuous on B_α with generator $(L, B_{1+\alpha})$.

Proposition 63 $S_t(B) \subseteq B_\alpha$ for any $t, \alpha > 0$; and

$$\| (\lambda - L)^\alpha S_t x \| \leq \frac{C(\alpha, \lambda)}{t^\alpha} \|x\| \quad \forall x \in B, \lambda > a$$

Proof cf. e.g. [Hairer Ch. 4]. $\square$ (Exercise: Proof for L self-adj.)

Corollary 64 $\forall \alpha \in (0,1]$ $\exists C_\alpha \in (0, \infty)$:

(i) $\| S_t x - x \| \leq C_\alpha t^\alpha \|x\|_\alpha \quad \forall x \in B_\alpha, t \in (0,1]$.

(ii) $\| S_t x - x \|_\gamma \leq C_\alpha t^\alpha \|x\|_{\alpha+\gamma} \quad \forall x \in B_{\alpha+\gamma}, t \in (0,1], \gamma \geq 0$

- trade-off: time regularity $\leftrightarrow$ spatial regularity
- $O(t)$ holds for $x \in B_1 = \mathrm{Dom}(L)$

Proof 1) W.l.o.g. $a < 0$, otherwise consider $\widehat{S}_t := e^{-(1+a)t} S_t$,

this is analytic semigroup with $\hat{a} = a - (1+a) = -1$.

2) $a < 0 \Rightarrow \lambda := 0 \in \rho(L) \Rightarrow$

$$\| S_t x - x \| = \left\| \int_0^t S_r L x \, dr \right\| \stackrel{\downarrow}{=} \left\| \int_0^t (-L)^{1-\alpha} S_r (-L)^\alpha x \, dr \right\|$$

operators commute by definition of $(-L)^\alpha$

$$\leq \int_0^t \| (-L)^{1-\alpha} S_r \| \, \|x\|_\alpha \, dr \stackrel{P.63}{\leq} C(\alpha, 0) \int_0^t r^{\alpha-1} dr \, \|x\|_\alpha$$

3) $\|S_t x - x\|_\gamma = \|(\lambda-\mathcal{L})^\gamma (S_t x - x)\| = \|S_t(\lambda-\mathcal{L})^\gamma x - (\lambda-\mathcal{L})^\gamma x\| \le C_\alpha t^\alpha \|(\lambda-\mathcal{L})^\gamma x\|_\alpha = C_\alpha t^\alpha \|x\|_{\gamma+\alpha}$ $\overset{4.27}{}$ $\square$

4.3. Time and space regularity for linear SPDE

$$(*) \quad dX_t = LX_t\,dt + \sigma\,dW_t \quad , \quad X_0 = x_0$$

L generator of analytic semigroup (S_t) on B, B Hilbert space

(W_t) cyl. Wiener over H, $\sigma \in L(H,B)$

$$\int_0^t \|S_r \sigma\|^2_{L^2(H,B)}\,dr < \infty \quad \forall t > 0$$

$\Rightarrow X_t = S_t x_0 + \int_0^t S_{t-s}\sigma\,dW_s$ unique weak sol. of $(*)$

THEOREM 65 (Spatial regularity) Let $\alpha, \beta \ge 0$ s.t. $\beta \le \frac{1}{2} + \alpha$.

If $\sigma \in L(H, B_\alpha)$ and $\|(\lambda-L)^{-\beta}\|_{L_2(B,B)} < \infty$ for some $\lambda \ge 0$, $\lambda \in \rho(L)$,

then $X_t \in B_\gamma$ $\forall t \ge 0$ a.s. for any $\gamma \in [0, \frac{1}{2} + \alpha - \beta)$.

Proof $(*)$ has solution in B_γ

(w.r.t. generator $(L, B_{1+\gamma})$ of $S_t|_{B_\gamma}$) provided

$$(**) \quad \int_0^t \|S_r \sigma\|^2_{L_2(H,B_\gamma)}\,dr < \infty \quad \forall t > 0.$$

In this case, by uniqueness, $X_t \in B_\gamma$ $\forall t \geq 0$ a.s.
Verification of (**):

$$\|S_r\sigma\|_{L_2(H,B_\gamma)} = \|(\lambda-L)^\gamma S_r\sigma\|_{L_2(H,B)} \leq \|(\lambda-L)^\gamma S_r\|_{L_2(B_\alpha,B)} \, \|\sigma\|_{L(H,B_\alpha)}$$

$$\|(\lambda-L)^\gamma S_r\|_{L_2(B_\alpha,B)} = \|\underbrace{(\lambda-L)^\gamma S_r(\lambda-L)^{-\alpha}}_{=(\lambda-L)^{\gamma-\alpha}S_r}\|_{L_2(B,B)}$$

$$\lesssim \|(\lambda-L)^{-\beta}\|_{L_2(B,B)} \cdot \underbrace{\|(\lambda-L)^{\beta+\gamma-\alpha}S_r\|_{L(B,B)}}_{\overset{\text{Prop 63}}{\leq} \text{const.}\cdot\left(r^{\alpha-\beta-\gamma} \vee 1\right)}$$

$\Rightarrow$ (**) holds provided $\alpha-\beta-\gamma > -1/2$

$\square$

Example 1) $dX_t = \Delta X_t\,dt + (-\Delta)^{-\alpha}dW_t$ on $[0,1]^d$

Δ Dirichlet Laplacian on $B = L^2([0,1]^d)$,

(W_t) cyl. Wiener process over $H = B$, $\sigma = (-\Delta)^{-\alpha} \in L(H, B_\alpha)$

$B_\gamma = H_{2\gamma}$, cf. Section 2.5

$i: H_r \to H_s$ Hilbert-Schmidt for $s < r - \frac{d}{2}$, cf. Thm. 15

$$\|(-\Delta)^{-\beta}\|_{L_2(B,B)} = \|I\|_{L_2(B,B_{-\beta})} < \infty$$

provided $2\beta > d/2$, i.e. $\beta > d/4$

Hence $X_t \in B_\gamma$ for $\gamma \in [0, \frac{2-d}{4} + \alpha)$

$\underline{d=1,\ \alpha=0}$: $X_t \in H_{2\gamma}$ a.s. $\forall \gamma < 1/4$

$\underline{d=2}$: $X_t \in \bigcup_{\varepsilon>0} H_\varepsilon$ a.s. provided $\alpha > 0$

$\underline{d>2}$: $X_t \in \bigcup_{\varepsilon>0} H_\varepsilon$ a.s. provided $\alpha > \frac{d-2}{4}$

2) Δ with periodic boundary conditions: Similarly for

$$dX_t = \Delta X_t dt + (1-\Delta)^{-\alpha} dW_t$$

<u>THEOREM 66 (Space-time regularity)</u> Setup as in Thm. 65

Then $t \mapsto X_t$, $\mathbb{R}_+ \to B_\gamma$, $\gamma \in [0, \frac{1}{2}+\alpha-\beta)$, is almost surely δ-Hölder continuous w.r.t. $\|\cdot\|_\gamma$ for any $\delta \in (0; \min(\frac{1}{2}, \frac{1}{2}+\alpha-\beta-\gamma))$

<u>Example</u> Stoch. heat equation in $d=1$:

δ-Hölder continuous in $H_{1/2-\delta}$ for any $\delta\in(0,1/2)$

<u>Proof</u> via Kolmogorov-Čentsov (Cor. 22):

$$X_t = \underbrace{S_{t-s}X_s}_{\mathcal{F}_s\text{-measurable}} + \underbrace{\int_s^t S_{t-r}\sigma\,dW_r}_{\text{indep. of } \mathcal{F}_s} \quad \forall\, 0\le s\le t \quad \text{a.s.}$$

independent, uncorrelated

$$\Rightarrow E\left[\|X_t-X_s\|_\gamma^2\right] = E\left[\|S_{t-s}X_s - X_s\|_\gamma^2\right] + E\left[\left\|\int_s^t S_{t-r}\sigma\,dW_r\right\|_\gamma^2\right]$$

$$\overset{\text{Cor. 64}}{\underset{\text{Itô isometry}}{\le}} C_\delta\,(t-s)^{2(\delta\wedge 1)}\overbrace{E\left[\|X_s\|^2_{\underbrace{\delta+\gamma}_{<\frac{1}{2}+\alpha-\beta}}\right]}^{<\infty} + E\left[\int_s^t \underbrace{\|S_{t-r}\sigma\|^2_{L_2(H,B_\gamma)}}_{\le \text{const.}\,(t-r)^{2(\alpha-\beta-\gamma)}}\,dr\right]$$

cf. Proof above

$$\le \text{const.}\,(t-s)^{2\delta} \quad \text{for any } \delta \text{ as above}$$

$$\Rightarrow \quad E\left[\|X_t - X_s\|_\gamma^{2n}\right] \le \text{const.} \cdot E\left[\|X_t - X_s\|_\gamma^2\right]^n$$

(X_t) Gaussian process

$$\le \text{const.}\ (t-s)^{2\delta n}$$

Kolm.-Čentsov

$\Rightarrow$ (X_t) a.s. $\tilde{\delta}$-Hölder in B_γ for any

$$\tilde{\delta} < \delta - \frac{1}{2n}, \quad n \in \mathbb{N}, \quad \delta < \min\left(\frac{\alpha}{2}, \frac{1}{2} + \alpha - \beta - \gamma\right)$$

$\square$

Example: Stochastic heat equation on $\mathbb{R}^1$

(*) $du_t = \frac{1}{2}\Delta u_t\,dt + dW_t$, W_t cyl. Wiener over $L^2(\mathbb{R},dx)$,

$\Delta = \left(\frac{d^2}{dx^2}, H^2(\mathbb{R},dx)\right)$ self-adjoint on $L^2(\mathbb{R},dx)$

$$P_t = e^{t\Delta/2}, \quad (P_t f)(x) = \int \underbrace{p_t(x,y)}_{=(2\pi t)^{-1/2} e^{-|x-y|^2/(2t)}} f(y)\,dy$$

(**) $$u_t = P_t u_0 + \int_0^t P_{t-r}\,dW_r$$

However: P_t is not Hilbert-Schmidt on $L^2(\mathbb{R},dx)$:

$$\|P_t\|_{L_2}^2 = \mathrm{tr}(P_t^* P_t) = \mathrm{tr}(P_{2t}) = \int_{\mathbb{R}} \underbrace{p_{2t}(x,x)}_{=p_{2t}(0,0)}\,dx = \infty.$$

$$L^2(\mathbb{R},dx) \to L^2(\mathbb{R},\omega(x)dx) =: \tilde{B},$$

ω growing sufficiently fast at $\pm\infty$

$\Rightarrow \int_0^t \|P_r\|_{L_2(\tilde{B},\tilde{B})}\,dr < \infty \Rightarrow \exists\, u \in C(\mathbb{R}_+, \tilde{B})$ a.s.

Actually,

$$u_t \in \underbrace{C_{pol}(\mathbb{R}_+)}_{} \quad \text{almost surely}$$

cont. fcts. with at most polynomial growth as $|x| \to \infty$

$$\underline{\underline{u_t(x)}} = \langle \delta_x, u_t \rangle = \langle \delta_x, P_t u_0 \rangle + \int_0^t \langle \delta_x, P_{t-r} dW_r \rangle$$

(****) $$= \underline{\underline{\int_{\mathbb{R}} p_t(x,y)\, u_0(y)\, dy + \int_0^t \langle p_{t-r}(x,\cdot), dW_r \rangle}} \quad \forall t \geq 0 \text{ a.s.}$$

Space-time regularity of $u_t(x)$? Wlog $u_0 = 0$.

$\Rightarrow \quad E[u_t(x)] = 0 \quad \forall t, x$

$$\operatorname{Cov}(u_t(x), u_s(y)) \overset{s \leq t}{=} E\Big[\int_0^t \langle p_{t-r}(x,\cdot), dW_r \rangle \int_0^t I_{\{r \leq s\}} \langle p_{s-r}(y,\cdot), dW_r \rangle\Big]$$

$$\overset{\text{Itô isometry}}{=} \int_0^t I_{\{r \leq s\}} \big(p_{t-r}(x,\cdot), p_{s-r}(y,\cdot) \big)_{L_2(L^2(\mathbb{R},dx),\mathbb{R})} \, dr$$

$$= \int_0^s \int_{\mathbb{R}} p_{t-r}(x,z)\, p_{s-r}(y,z)\, dz\, dr = \int_0^s p_{t+s-2r}(x,y)\, dr$$

$$\mathrm{Cov}\left(u_t(x), u_s(y)\right) = \frac{1}{2}\int_{|t-s|}^{t+s} p_r(x,y)\, dr$$

$$= \frac{1}{\sqrt{8\pi}} \int_{|t-s|}^{t+s} \frac{1}{\sqrt{r}}\, e^{-\frac{|x-y|^2}{2r}}\, dr \qquad \forall s,t\geq 0,\ x,y\in\mathbb{R}$$

Time-regularity:

$$\mathrm{Cov}\left(u_t(x), u_s(x)\right) = \frac{1}{\sqrt{2\pi}}\left(\sqrt{t+s} - \sqrt{|t-s|}\right)$$

$$E\left[\left(u_t(x)-u_s(x)\right)^2\right] = \frac{1}{\sqrt{2\pi}}\left(\sqrt{2t}+\sqrt{2s}-2\sqrt{t+s}+2\sqrt{|t-s|}\right)$$

$$= O\left(\sqrt{|t-s|}\right)$$

$$\overset{\text{Kolm.-Čentsov}}{\Longrightarrow} \quad t\mapsto u_t(x) \text{ a.s. } \alpha\text{-Hölder-contin. } \underline{\underline{\forall \alpha < \tfrac{1}{4}}}$$

NOT 1/2 !

Spatial regularity:

$$\mathrm{Cov}\left(u_t(x), u_t(y)\right) = \frac{1}{\sqrt{8\pi}} \int_0^{2t} \frac{1}{\sqrt{r}}\, e^{-\frac{|x-y|^2}{2r}}\, dr$$

singular at 0

$$\overset{z=\frac{|x-y|^2}{2r}}{=} \frac{|x-y|}{4\sqrt{\pi}} \int_{\frac{|x-y|^2}{4t}}^{\infty} z^{-3/2} e^{-z}\, dz$$

$$\overset{\text{ibp}}{=} \frac{|x-y|}{2\sqrt{\pi}}\left[\left(\frac{|x-y|^2}{4t}\right)^{-1/2} - \ldots \int_{\frac{|x-y|^2}{4t}}^{\infty} z^{-1/2} e^{-z}\, dz\right]$$

$$= \sqrt{\frac{t}{\pi}} - \frac{|x-y|}{2} + |x-y| \cdot O\left(\frac{|x-y|}{\sqrt{t}}\right)$$

divergent as $t \to \infty$ (does not occur for $\mathbb{R} \to [0,1]$ with Dirichlet boundary conds.)
$Var(u_t(x)) \sim \sqrt{t/\pi}$

$$E\left[(u_t(x) - u_t(y))^2\right] = 2\sqrt{\frac{t}{\pi}} - 2(\ \dots\ )$$

$$= |x-y| \cdot \left(1 + O\left(|x-y|/\sqrt{t}\right)\right)$$

$|x-y| < \sqrt{t}$: spatial regularity of $u_t(x)$

$\hat{=}$ time regularity of standard BM

$x \mapsto u_t(x)$ a.s. α-Hölder cont. on compact intervals

for any $\underline{\alpha < 1/2}$

<u>Exercise</u>: Same computation for $du_t = \frac{1}{2}\Delta u_t\, dt - u_t + dW_t$. $t \to \infty$?

<u>Time-space regularity</u> $(t,x) \mapsto u_t(x)$

is a.s. $1/4$-Hölder for $(t,x) \in [0, t_0] \times [x_0, x_1]$

Proof via Kolm.-Čentsov 2D.

<u>Remark</u>: Time regularity $\alpha = \frac{1}{2} - \varepsilon$ holds in space H_δ but not in C_0!

5. Semilinear SPDE

B sep. Banach space, H sep. Hilbert space

$$(*) \quad dX_t = LX_t\,dt + F(X_t)\,dt + \sigma\,dW_t\,, \quad X_0 = x_0\,,$$

L generator of C_0 semigroup $(S_t)_{t\geq 0}$ on B,

$F : \underbrace{D(F)}_{\text{linear subspace}} \subseteq B \to B$ measurable

(W_t) cylindrical Wiener over H, $\sigma \in L(H,B)$

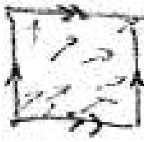
Example 1) Reaction-diffusion equations

$B = C(U, \mathbb{R}^k)$, $U \subseteq \mathbb{R}^d$, $L = \Delta$, $f : \mathbb{R}^k \to \mathbb{R}^k$

$F(u) = f \circ u$, i.e. $F(u)(s) = f(u(s))$ $\forall s \in U$

"$\frac{\partial X}{\partial t}(t,s) = \underbrace{(\Delta_s X)(t,s)}_{\text{diffusion}} + \underbrace{f(X(t,s))}_{\text{reaction}} + \text{noise}$"

$X^i(t,s)$ = concentration of component i at time t and position s

2) <u>Navier-Stokes</u>  div $u = 0$

motion of idealized incompressible fluid, $B = L^2(T, \mathbb{R}^2)$

$u(t,s) \in \mathbb{R}^2$ instantaneous velocity at time t and position $s \in T$,

$T = [0,1]^2/\sim$ two-dim. torus

$$\underset{\substack{\text{viscosity,}\\ \text{constant} > 0}}{\frac{\partial u}{\partial t} = \nu} \Delta u + (u \cdot \nabla) u - \underset{\substack{p(t,s) \text{ pressure,}\\ \text{determined by div } u = 0}}{\nabla p} + \text{noise}, \quad \underbrace{\operatorname{div} u = 0}_{\text{incompressibility}}$$

$\Pi :=$ orth. proj. in $L^2(T, \mathbb{R}^2)$ onto divergence-free vector fields

$$\rightarrow \quad \frac{\partial u}{\partial t} = \nu \Delta u + \Pi (u \cdot \nabla) u + \text{noise}$$

$$du_t = \nu \Delta u_t + F(u_t) + \sigma \, dW_t$$

with $F(u) = \Pi (u \cdot \nabla) u$

Assumption: $\exists Y_t = \int_0^t S_{t-s}\sigma \, dW_s \in C(\mathbb{R}_+, B)$ a.s. (solution of linear equation with $x_0 = 0$.) §3

§.1. Existence & uniqueness of local solutions

$\mathcal{F}_t = \sigma(W_s \mid 0 \le s \le t)$

$T : \Omega \to [0,\infty]$ $(\mathcal{F}_t)$ stopping time, i.e., $\{T \le t\} \in \mathcal{F}_t$ $\forall t \ge 0$.

Def. $X_t : \Omega \to B$, $t < T$, is called a local mild solution of $(*)$ iff almost surely, $X_t \in D(F)$ and

$$(**) \quad X_t = S_t x_0 + \int_0^t S_{t-s} F(X_s)\, ds + Y_t \quad \forall t < T$$

Remark $\dim(B) < \infty$, $S_t = e^{tL}$:

$(X_t)_{t<T}$ local mild sol. $\overset{\text{variation of constants}}{\iff}$ (X_t) sol. of $(*)$ up to T

$B(0,r) := \{x \in B : \|x\| < r\}$

THEOREM 67 Suppose that $F|_{B(0,r)}$ is Lipschitz contin. for any $r \in (0,\infty)$. Then:

1) $\exists!$ local mild solution $(X_t)_{t<T}$ of $(*)$ s.t. almost surely, $t \mapsto X_t$ is continuous a.s. and $\limsup_{t \uparrow T} \|X_t\| = \infty$.

2) F globally Lipschitz on $B \Rightarrow T = \infty$ a.s.

Example 1) Reaction-diffusion equation: $F(x) = f \circ x$

$$\Rightarrow \|F(x)-F(y)\|_{\sup} = \sup_s |f(x(s)) - f(y(s))| \le L_r \|x-y\|$$

for $x, y \in B(0,r)$ provided f is Lipschitz on $B(0,r) \subseteq \mathbb{R}^k$ with constant L_r

$\Rightarrow$ Theorem applies if f loc. Lipschitz

2) Navier-Stokes: $F(u) = \Pi(u \cdot \underbrace{\nabla u}_{\text{unbounded lin. op.}})$ not loc. Lipschitz

However: $\|F(u)\|_{H^t} \le C \|u\|_{H^{t+1}} \dots$

<u>Proof of Theorem 67</u> via fixed point argument

(1) Fix $R>0$, $g \in C([0,t], B)$.

<u>Claim:</u> $M = M_{g,t} : C([0,t], B) \to C([0,t], B)$

$$(Mu)_r := \int_0^r S_{r-s} F(u_s)\, ds + g_r$$

is contractive on $A_R := \{u : \sup_{r \le t} \|u_r - g_r\| \le R\}$ for t small.

Proof: $\sup_{[0,t]} \|Mu - Mv\| \le t \cdot \underbrace{\sup_{r \in [0,t]} \|S_r\|}_{\substack{<\infty \text{ \& increasing} \\ \text{since } (S_r) \text{ strongly cont.}}} \cdot \underbrace{\sup_{r \in [0,t]} \|F(u_r) - F(v_r)\|}_{\le L \sup_{[0,t]} \|u-v\|}$

$$\le \frac{1}{2} \sup_{[0,t]} \|u - v\|$$

for $t \le t_0(g,R)$, $u, v \in A_R$

$\sup_{[0,t]} \|Mu - g\| \le \text{const} \cdot t \cdot \underbrace{\sup_{[0,t]} \|F(u_r)\|}_{\le \|F(g_r)\| + L \cdot \sup_{[0,t]} \|u-g\|} < R$

M_0

for $t \le t_1(g,R)$, $u \in A_R$

$\Rightarrow$ $M : A_R \to A_R$ contraction for $t = \min(t_0, t_1)$

$\overset{\text{Banach}}{\Longrightarrow}$ $\exists!$ fixed point u,

$$u_r = (M u)_r = \int_0^r S_{r-s} F(u_s)\, ds + g_r \quad \forall r \le t$$

(2) Application to SPDE (pathwise, ω fixed):

GOAL: $$X_t = S_t x_0 + \int_0^t S_{t-s} F(X_s)\, ds + Y_t \quad \forall t > 0$$

$$t \mapsto Y_t = \int_0^t S_{t-s} \sigma\, dW_s \in B \text{ continuous}$$

$(X_r)_{r \in [0,t]}$ solution up to t

$$X_{t+h} \stackrel{!}{=} S_{t+h} x_0 + \int_0^{t+h} S_{t+h-s} F(X_s)\, ds + Y_{t+h}$$

$$= \underbrace{S_h \left(S_t x_0 + \int_0^t S_{t-s} F(X_s)\, ds \right) + Y_{t+h}}_{=: g_h} + \int_0^h S_{h-r} F(X_{t+r})\, dr$$

solution exists for $h \in [0, \varepsilon(t)]$

$\rightsquigarrow$ can continue X_r to $[0, t + \varepsilon(t)]$

$\rightsquigarrow$ maximal solution up to explosion time T

(3) F globally Lipschitz $\Rightarrow$ M contraction on B for any $t \le t_0$, t_0 indep. of g

$\Rightarrow$ [illegible]

Semilinear SDE on B:

$$(*) \quad dX_t = LX_t\,dt + F(X_t)\,dt + \sigma\,dW_t, \quad X_0 = x_0$$

L gen. of C_0 semigroup (S_t) on B, $\sigma \in L(H,B)$,

$F: D(F) \subseteq B \to B$ measurable

Local mild solution:

$$(**) \quad X_t = S_t x_0 + \int_0^t S_{t-s} F(X_s)\,ds + \underbrace{\int_0^t S_{t-s}\sigma\,dW_s}_{=: Y_t} \quad \forall t < T$$

F locally Lipschitz on B

$\Rightarrow \exists!$ local mild solution up to explosion

not applicable to stoch. Navier Stokes !

Extension: Suppose (S_t) analytic semigroup,

$\|x\|_\alpha := \|(\lambda - L)^\alpha x\|$,

$$B_\alpha := \begin{cases} \operatorname{Dom}(\lambda - L)^\alpha & \text{for } \alpha \geq 0 \\ \overline{B}^{\|\cdot\|_\alpha} & \text{for } \alpha < 0 \end{cases}$$ interpolation spaces

<u>THEOREM 68</u> Let $\alpha \geq 0$. Suppose that:

(i) $Y \in C(\mathbb{R}_+, B_\alpha)$ a.s., $x_0 \in B_\alpha$

(ii) $\exists \gamma \geq 0$, $\delta \in [0,1)$ $\forall \beta \in [0,\gamma]$; $r \in (0,\infty)$ $\exists L_\beta(r)$

$$\|F(x) - F(y)\|_{\beta-\delta} \leq L_\beta(r) \|x-y\|_\beta \quad \forall x,y : \|x\|_\beta, \|y\|_\beta < r$$

(i.e. F extends to loc. Lipschitz map from B_β to $B_{\beta-\delta}$)

(iii) $\forall \beta \in [0,\gamma]$ $\exists C, n$: $\|F(x)\|_{\beta-\delta} \leq C \cdot (1+|x|^n)$, $L_\beta(r) \leq C \cdot (1+r^n)$

(polynomial growth)

Then $\exists!$ local mild solution $(X_t)_{t<T}$ up to explosion s.t.

$$X_t \in B_\beta \quad \forall t \geq 0, \ \beta < \beta_* := \min(\alpha, \gamma+1-\delta)$$

Remark: Here we only consider the case $\alpha = \gamma = 0$ (and thus $\beta = 0$)

<u>Example (2D-Navier Stokes)</u> $u : T \to \mathbb{R}^2$ divergence free

$$L = \Delta \quad , \quad F(u) = \Pi (u \cdot \nabla) u$$

Π: orth. proj. onto div. free v.f.

$$B := H^s, \ s > 1 \implies B_\alpha = H^{s+2\alpha} \quad \forall \alpha \in \mathbb{R}$$

<u>Fact</u>: $\|uv\|_{H^t} \leq \text{const.}\, \|u\|_{H^s} \|v\|_{H^r}$

provided $s, r > t \geq 0$ and $s + r > t + \frac{d}{2}$.

(Consequence of Sobolev embedding + Hölder,
cf. Hairer Thm. 6.25)

$$\overset{\substack{d=2 \\ s-1>t\geq 0}}{\Rightarrow} \|F(u)\|_{H^t} \leq \text{const.} \|u\|_{H^s} \|\nabla u\|_{H^{s-1}} \leq \text{const.}\, \|u\|^2_{H^s}$$

$$\|F(u) - F(v)\|_{H^t} \leq \|u\,(\nabla u - \nabla v)\|_{H^t} + \|(u-v)\cdot \nabla v\|_{H^t}$$

$$\leq \text{const.}\, \left(\|u\|_{H^s} + \|v\|_{H^s}\right) \|u - v\|_{H^s}$$

Hence assumptions of Thm. 6.8 are satisfied for $B = H^s$, $\gamma = 0$, $\delta > \frac{1}{2}$

$$\gamma = 0 \Rightarrow \beta = 0 \Rightarrow B_\beta = B = H^s, \quad B_{\beta - \delta} \underset{\substack{\uparrow \\ \delta := \frac{1}{2} + \varepsilon}}{=} H^{\overbrace{s-1-2\varepsilon}^{t}}$$

Thus $\exists!$ local mild solution $X_t \in H^s$.

<u>Proof of Theorem 68 (Sketch):</u>

based on $\|S_t x\|_\beta \le C_{\alpha,\beta}\, t^{\alpha-\beta} \|x\|_\alpha \quad \forall \alpha < \beta$

(i) <u>Existence</u>: Fix $g \in C([0,t], B)$

$$(Mu)_r := \int_0^r S_{r-s} F(u_s)\, ds + g_r$$

$$\Rightarrow \sup_{[0,t]} \|Mu - Mv\| \le C_{-\delta,0} \int_0^t s^{-\delta}\, ds \cdot \sup_{[0,t]} \|F(u)-F(v)\|_{-\delta}$$

$$\le \text{const.} \cdot t^{1-\delta} \cdot \sup_{[0,t]} \|u-v\|$$

provided $u, v \in B(0,r)$ on $[0,t]$

$\xRightarrow{\text{as above}}$ $\exists!$ fixed point for t suff. small

$\xRightarrow{\text{iteration}}$ $\exists!$ max. local mild solution $(X_t)_{t<T}$

(ii) <u>Regularity</u>: $r < t$.

$$X_t = S_t x_0 + \int_0^t S_{t-s} F(X_s)\, ds + Y_t$$

$$= S_{t-r} X_r + \int_r^t S_{t-s} F(X_s)\, ds + (Y_t - S_{t-r} Y_r)$$

Now choose $a \in (0,1)$, $r = at$:

$$X_t = S_{(1-a)t} X_{at} + \int_{at}^{t} S_{t-s} F(X_s) ds + \overbrace{\left(Y_t - S_{(1-a)t} Y_{at} \right)}^{=: Y_t^{a}}$$

$\varepsilon \in (0, 1-\delta)$:

$$\|X_t\|_\varepsilon \le C t^{-\varepsilon} \|X_{at}\| + \|Y_t^a\|_\varepsilon + C \int_{at}^{t} (t-s)^{-(\varepsilon+\delta)} (1 + \|X_s\|^n) ds$$

$\rightsquigarrow$ regularity w.r.t. $\|\cdot\|_\varepsilon$ provided $Y_t \in D_\varepsilon$ (i.e. $\varepsilon < \alpha$)

$\rightsquigarrow$ now iterate to get $\square$

5.2. Global solutions

a) Reaction-diffusion equation: $f: \mathbb{R}^d \to \mathbb{R}^d$ loc. Lip., e.g. $f(u) = u - u^3$

(*) $du_t = \Delta u_t \, dt + f \circ u_t \, dt + \sigma \, dW_t$

$u : \mathbb{R}_+ \to C(T^n, \mathbb{R}^d) =: B$, $T^n = [0,1]^n / \sim$ n-torus

$\Delta = (\Delta, H^2(T^n; \mathbb{R}^d))$ $\hat{=}$ self-adj. realiz. with periodic b.c. on $[0,1]^n$

Assumption: $\exists \, Y_t = \int_0^t P_{t-s} \sigma \, dW_s$; $Y \in C(\mathbb{R}_+, B)$ a.s.

where $P_t = e^{t\Delta}$ heat semigroup on vector fields $T^n \to \mathbb{R}^d$

$\Rightarrow \exists!$ maximal cont. mild sol. $(u_t)_{t < T}$, T stopping time

Conditions for $T \equiv \infty$ (non-explosion)?

Standard approach for SDE: stoch. Lyapunov functions

e.g. $\varphi \geq 0$ s.t. $e^{-\lambda t} \varphi(u_t)$ local supermart. up to T for $\lambda > 0$

(ok if $\mathcal{L} \varphi \leq \lambda \varphi$)

$\Rightarrow$ bound for $E[e^{-\lambda(t \wedge T_n)} \varphi(u_{t \wedge T_n})]$, $T_n \nearrow T$

$\Rightarrow \sup T_n = \infty$ a.s. provided $\varphi(x) \to \infty$ as $\|x\| \to \infty$.

Here: Consider $\varphi(u) = \sup_{x \in T^n} V(u(x))$,

[pathwise argument instead of martingale estimate]

<u>THEOREM 69</u> Suppose that $\exists V \in C^2(\mathbb{R}^d, \mathbb{R}_+)$ convex s.t.

(i) $V(x) \to \infty$ as $|x| \to \infty$

(ii) $\forall r > 0 \ \exists C_r \in (0,\infty): \ f(x+y)\cdot\nabla V(x) \leq C_r V(x) \ \forall |y| \leq r$

Then $T = \infty$ a.s.

<u>Remark</u> V convex, $(P_t u)(x) = \int p_t(x,dy)\, u(y)$ with stoch. kernel p_t

$$\Rightarrow \underbrace{\sup V(P_t u)}_{\varphi(P_t u)} \leq \underbrace{\sup V(u)}_{\varphi(u)} \quad \forall u \in B = C(T^n, \mathbb{R}^d)$$

by Jensen: $V\left(\int p_t(x,dy)\, u(y)\right) \leq \int p_t(x,dy)\, V(u(y)) \leq \sup V(u)$

<u>Proof of Theorem</u> $w_t := u_t - Y_t$, $t < T$, fix $\omega \in \Omega$ (pathwise)

$$\Rightarrow w_t = P_t w_0 + \int_0^t P_{t-s} \underbrace{f \circ u_s}_{=: X_s} \, ds$$

$=: X_s$ continuous $\mathbb{R}_+ \to B$

Claim $\forall r \in (0,\infty),\ t \geq 0$:

$$\|Y_t\| \leq r \implies \limsup_{h \downarrow 0} \frac{\varphi(w_{t+h}) - \varphi(w_t)}{h} \leq C_r\, \varphi(w_t)$$

Consequence:

Let $t_0 > 0$: $\sup_{t < t_0} \|Y_t\| < \infty \implies \underbrace{\varphi(w_t)}_{= \sup V(w_t)}$ bounded for $t < t_0 \wedge T$

$$\overset{(i)}{\implies} \sup_{t < t_0 \wedge T} \|w_t\| < \infty \implies \sup_{t < t_0 \wedge T} \|u_t\| < \infty \implies t_0 < T$$

Proof of claim (Sketch).

$$w_{t+h} = P_h w_t + \int_0^h P_{h-r} \overbrace{X_{t+r}}^{= X_t + o(h)} \, dr$$

$$= P_h(w_t + h X_t) + \underbrace{\int_0^h (P_{h-r} X_{t+r} - P_h X_t)\, dr}_{= o(h) \text{ w.r.t. } \|\cdot\|_B}$$

$$\implies \varphi(w_{t+h}) \leq \varphi(P_h(w_t + h X_t)) + o(h)$$

$$\overset{\text{Rem.}}{\leq} \varphi(w_t + h X_t) + o(h)$$

$$= \sup \underbrace{V(w_t + h X_t)}_{\overset{\text{Taylor}}{=} V(w_t) + h X_t \cdot \nabla V(w_t) + o(h)} + o(h)$$

$$\overset{(ii)}{\leq} \quad \varphi(w_t) + h\, C_r\, \varphi(w_t) + o(h)$$

$\uparrow$

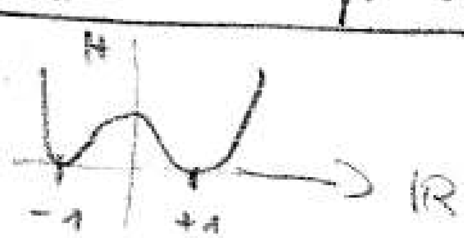
$X_t = f(w_t + Y_t)$ provided $\|Y_t\| \leq r$

$\square$

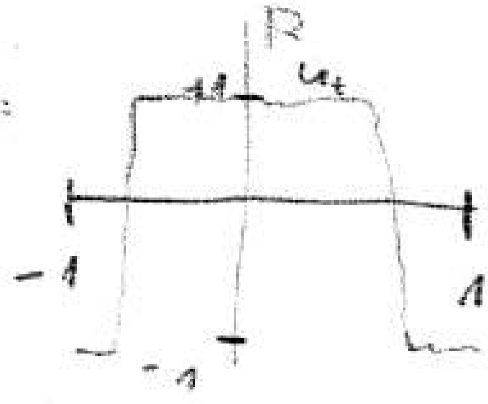
<u>Example: Stochastic Allen Cahn equation</u> $B = C([-1,1], \mathbb{R})$

$H(u) := \frac{1}{\varepsilon}(u^2-1)^2,\ \varepsilon > 0$

$f(u) = -H'(u)$



<u>$d=1$:</u>



preferred states ± 1 (phases)

phase boundaries

$\varepsilon \downarrow 0$: sharp interfacial limit

Theorem applies with $\sigma = I_{L^2}$, $V(u) = u^2$

$$du = \Delta u\, dt + \frac{4}{\varepsilon}(u - u^3)\, dt + \sigma\, dW$$

b) 2D-Navier Stokes: $u: \mathbb{T}^2 \to \mathbb{R}^2$, $\operatorname{div} u = 0$

$$(*) \quad du = \Delta u \, dt + \Pi (u \cdot \nabla) u \, dt + \sigma dW$$

$v := \operatorname{rot} u = \partial_1 u_2 - \partial_2 u_1$, $u = kv$ $(\operatorname{div} u = 0)$
"vorticity"

$$(**) \quad dv = \Delta v \, dt + \underbrace{(kv \cdot \nabla) v}_{=: F(v)} \, dt + \tilde{\sigma} dW$$

$=: F(v)$, loc. Lip. $L^2 \to H^{-1-\varepsilon}$

$$w_t := v_t - Y_t, \quad Y_t = \int_0^t e^{(t-s)\Delta} \tilde{\sigma} \, dW_s$$

$$(***) \quad \frac{dw}{dt} = \Delta w + F(w + Y)$$

$$\Rightarrow \frac{d}{dt} \|w\|_{L^2}^2 \leq -2 \|\nabla w\|_{L^2}^2 - 2 \underbrace{Y \cdot F(w+Y)}$$

$$|\cdot| \overset{(\dots)}{\leq} \|Y\|_{H^{1/2}} \|w+Y\|_{H^{1/2}}^2$$

$$\leq \dots \leq 8 \|Y\|_{H^{1/2}}^2 \|w\|_{L^2}^2 + 2 \|Y\|_{H^{1/2}}^3$$

$\Rightarrow$ Global existence whenever $Y \in C(\mathbb{R}_+, H^{1/2})$