

“Markov Processes”, Problem Sheet 1.

Please hand in your solutions before 10 am on Monday, October 19.

The deadline for the following problem sheets will be on fridays at 10 am.

1. (Reflected Random Walks and Metropolis algorithm). Let π be a probability kernel on a measurable space $(S, \mathcal{B})$. A probability measure μ satisfies the *detailed balance condition* w.r.t. π iff for any $A, B \in \mathcal{B}$,

$$(\mu \otimes \pi)(A \times B) = (\mu \otimes \pi)(B \times A).$$

- a) Prove that a probability measure μ which satisfies detailed balance w.r.t. π is also invariant w.r.t. π , and that for all $f, g \in \mathcal{F}_b(S)$,

$$\int f \pi g \, d\mu = \int \pi f g \, d\mu.$$

- b) Let $S \subset \mathbb{R}^d$ be a bounded measurable set. We define a Markov chain $(X_n)_{n \in \mathbb{Z}_+}$ by

$$X_0 = x_0 \in S, \quad X_{n+1} = X_n + W_{n+1} \cdot 1_S(X_n + W_{n+1}),$$

where $W_i : \Omega \rightarrow \mathbb{R}^d$ are i.i.d. random variables. Suppose that the law of W_1 is absolutely continuous with a strictly positive density satisfying $f(x) = f(-x)$. Prove that the uniform distribution on S is invariant w.r.t. the transition kernel of the Markov chain (X_n) .

- c) Let $\mu(dx) = \mu(x) dx$ be a probability measure with strictly positive density on $\mathbb{R}^d$. Show that the process (X_n) , defined by the following algorithm, is a time-homogeneous Markov chain, identify its transition kernel π , and show that μ is invariant for π .

Random walk Metropolis algorithm

- 1.) Set $n := 0$ and choose some arbitrary point $X_0 \in \mathbb{R}^d$.
- 2.) Set $Y_{n+1} := X_n + W_{n+1}$, and draw independently $U_{n+1} \sim \text{Unif}(0, 1)$.
- 3.) If $U_{n+1} < \mu(Y_{n+1})/\mu(X_n)$ then set $X_{n+1} := Y_{n+1}$, else set $X_{n+1} := X_n$.
- 4.) Set $n := n + 1$ and go to Step 2.

2. (Markov properties).

- a) Let $(X_t)_{t \in \mathbb{R}_+}$ be a Markov process with transition function $(p_{s,t})$ and state space S on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Show that for any $t_0 \in \mathbb{R}_+$, the time-space process $\hat{X}_t = (t_0 + t, X_{t_0+t})$ is a time-homogeneous Markov process with state space $\mathbb{R}_+ \times S$ and transition function

$$\hat{p}_t((s, x), \cdot) = \delta_{s+t} \otimes p_{s, s+t}(x, \cdot).$$

- b) Let $(X_t)_{t \in \mathbb{R}_+}$ be an arbitrary stochastic process on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Show that the *historical process* $\hat{X}_t = (X_s)_{s \in [0, t]}$ is an $(\mathcal{F}_t^X)$ Markov process.
- c) Let (B_t) be a standard Brownian motion on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and let

$$X_t = \int_0^t B_s ds.$$

Show that (X_t) is not a Markov process, but the two component process (B_t, X_t) is a time-homogeneous Markov process.

- d) Determine the transition function of this Markov process.