Institut für Angewandte Mathematik
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Andreas Eberle, Francis Lörler

"Markov Processes", Problem Sheet 9

Please hand in your solutions by Friday, December 12, 13.00.

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1. (Semigroups, resolvents and generators).

a) Suppose that $(P_t)_{t\geq 0}$ is a strongly continuous contraction semigroup on a Banach space E with generator $(L, \mathrm{Dom}(L))$. Show that $G_\alpha f = \int_0^\infty e^{-\alpha t} P_t f \, \mathrm{d}t$ is a strongly continuous contraction resolvent, and

$$G_\alpha(\alpha I - L)f \;=\; f \qquad \text{for any } f \in \mathrm{Dom}(L).$$

b) State the conditions in the Hille-Yosida Theorem, and verify that under these conditions, $G_\alpha = (\alpha I - L)^{-1}$ defines a strongly continuous contraction resolvent.

2. (Uniform motion to the right). Consider the deterministic Markov process $((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}})$ on $\mathbb{R}$ given by $X_t = x + t$ $\mathbb{P}_x$-almost surely.

a) Show that the transition semigroup $(P_t)_{t\geq 0}$ is strongly continuous both on $\hat{C}(\mathbb{R})$ and on $L^2(\mathbb{R}, \mathrm{d}x)$.

b) Prove that the generator on $\hat{C}(\mathbb{R})$ is given by

$$Lf \;=\; f', \qquad \mathrm{Dom}(L) \;=\; \{f \in C^1(\mathbb{R}) : f, f' \in \hat{C}(\mathbb{R})\}.$$

c) Show that the generator on $L^2(\mathbb{R}, \mathrm{d}x)$ is given by

$$Lf \;=\; f', \qquad \mathrm{Dom}(L) \;=\; H^{1,2}(\mathbb{R}, \mathrm{d}x).$$

3. (Strong continuity of transition semigroups of Markov processes on L^p spaces). Suppose that $(p_t)_{t\geq 0}$ is the transition function of a *right-continuous*, time-homogeneous Markov process $((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in S})$, and $\mu \in \mathcal{M}_+(S)$ is a sub-invariant measure.

a) Show that for every $f \in C_b(S)$ and $x \in S$,

$$(p_t f)(x) \to f(x) \qquad \text{as } t \downarrow 0.$$

b) Now let f be a non-negative function in $C_b(S) \cap \mathcal{L}^1(S, \mu)$ and $p \in [1, \infty)$. Show that as $t \downarrow 0$,

$$\int |p_t f - f| \, \mathrm{d}\mu \ \to \ 0\,, \qquad \text{and hence} \qquad p_t f \to f \text{ in } L^p(S, \mu).$$

Hint: You may use that $|x| = x + 2x^-$.

c) Conclude that (p_t) induces a strongly continuous contraction semigroup of linear operators on $L^p(S, \mu)$ for every $p \in [1, \infty)$.

4. (Ornstein-Uhlenbeck process). The transition semigroup of the Ornstein-Uhlenbeck process on $\mathbb{R}$ is given by

$$(p_t f)(x) \ = \ (2\pi)^{-1/2} \int f\left(e^{-t}x + \sqrt{1 - e^{-2t}}\, y\right) \, e^{-y^2/2} \, \mathrm{d}y \qquad \text{for } f \in \mathcal{F}_b(\mathbb{R}).$$

a) Show that the standard normal distribution γ is invariant.

b) Denote by C^2_{pol} the space of twice continuously differentiable functions on $\mathbb{R}$ such that f, f' and f'' grow at most polynomially at infinity. Let L denote the generator on $L^2(\mathbb{R}, \gamma)$. Show that $C^2_{\text{pol}} \subset \text{Dom}(L)$ and

$$(Lf)(x) \ = \ f''(x) - xf'(x) \qquad \text{for any } f \in C^2_{\text{pol}}.$$

c) Show that p_t preserves polynomials. Hence conclude that C^2_{pol} is a core for the generator.