Institut für Angewandte Mathematik
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Andreas Eberle, Francis Lörler

"Markov Processes", Problem Sheet 4

Please hand in your solutions by Friday, November 7, 13.00.

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1. (Martingales and boundary value problems). Let $(X_n, \mathbb{P}_x)$ be a canonical time-homogeneous Markov chain with measurable state space $(S, \mathcal{B})$ and generator $\mathcal{L}$, and let $T = \inf\{n \geq 0 : X_n \notin D\}$ denote the first exit time from a subset $D \in \mathcal{B}$.

a) Suppose that $v : S \to \mathbb{R}$ is a non-negative measurable function satisfying

$$\mathcal{L}v \;\leq\; -c \quad \text{in } D$$

for a non-negative function $c : D \to \mathbb{R}$. Prove that for every $x \in S$, the process

$$M_n \;=\; v(X_{n\wedge T}) + \sum_{i<n\wedge T} c(X_i) \tag{1}$$

is a non-negative supermartingale w.r.t. $\mathbb{P}_x$.

b) The second part of this exercise gives a direct proof of the uniqueness part in the Feynman-Kac formula for Markov chains. Let $w : S \to \mathbb{R}$ be a non-negative measurable function. Determine for which functions v the process

$$M_n \;=\; e^{-\sum_{k=0}^{n-1} w(X_i)} v(X_n)$$

is a martingale. Now suppose that for all $x \in S$, $T < \infty$ holds $\mathbb{P}_x$-almost surely, and let v be a bounded solution to the boundary value problem

$$\begin{aligned} (\mathcal{L}v)(x) &= (e^{w(x)} - 1)v(x) \quad \text{for all } x \in D, \\ v(x) &= f(x) \qquad\qquad\quad \text{for all } x \in S \setminus D. \end{aligned} \tag{2}$$

Show that

$$v(x) \;=\; \mathbb{E}_x\left[e^{-\sum_{k=0}^{T-1} w(X_k)} f(X_T)\right].$$

2. (Random walks on $\mathbb{Z}$). Suppose that $(X_n, \mathbb{P}_x)$ is a time-homogeneous Markov chain with state space $\mathbb{Z}$ and transition matrix π given by $\pi(x, x+1) = p$, $\pi(x, x) = r$, $\pi(x, x-1) = q$, where $p + q + r = 1$, $p > 0$, $q > 0$ and $r \geq 0$. Fix $a, b \in \mathbb{Z}$ with $a < b - 1$ and let

$$T \;=\; \inf\{n \geq 0 : X_n \notin (a, b)\}.$$

a) Prove that for every function $g : \{a+1, a+2, \ldots, b-1\} \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$, the following boundary value problem has a unique solution:

$$\begin{aligned} (\mathcal{L}u)(x) &= -g(x) \quad \text{for } a < x < b, \\ u(a) &= \alpha, \quad u(b) \;=\; \beta. \end{aligned} \tag{3}$$

b) Conclude that $\mathbb{E}_x[T] < \infty$ for all x.

c) How can the mean exit time be computed explicitly? Carry out the computation in the case where $p = q$ and $a = -b$.

Hint: Write down a system of equations for the "first derivative" $v(x) := u(x) - u(x-1)$.

3. (Recurrence on discrete state spaces). Let $(X_n, \mathbb{P}_x)$ be an irreducible time-homogeneous Markov chain with countable state space S.

a) Let $x, y \in S$. Show that if x is recurrent then

$$P_x[X_n = y \text{ infinitely often }] = 1,$$

and y is also recurrent.
Hint: Let $T^{(0)} := 0$ and let $0 < T^{(1)} < T^{(2)} < \dots$ denote the sequence of return times to the state x. Show that under $\mathbb{P}_x$, the events $A_k := \{X_n = y \text{ for some } n \in (T^{(k-1)}, T^{(k)})\}$, $k \in \mathbb{N}$, are independent with equal probability $p > 0$. Hence conclude that $\mathbb{P}_x$-almost surely, infinitely many of these events occur.

b) Prove that the following statements are all equivalent:

(i) There exists a finite recurrent set $A \subseteq S$.
(ii) There exists $x \in S$ such that the set $\{x\}$ is recurrent.
(iii) For any $x \in S$, the set $\{x\}$ is recurrent.
(iv) For any $x, y \in S$,

$$\mathbb{P}_x[X_n = y \text{ infinitely often }] = 1.$$

c) Show using Lyapunov functions that the simple random walk on $\mathbb{Z}^2$ is recurrent.
Hint: Consider for example the functions $V(x) = (\log(1 + |x|^2))^\alpha$ for $\alpha > 0$.

4. (Weak convergence of probability measures). Let μ_n $(n \in \mathbb{N})$ and μ be probability measures on the Borel σ-algebra over a metric space S. Prove that the following statements are equivalent:

(i) The sequence $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to μ.

(ii) For any bounded and Lipschitz continuous function $f : S \to \mathbb{R}$,

$$\int f\, d\mu_n \text{ converges to } \int f\, d\mu.$$

(iii) For any closed set $A \subseteq S$,

$$\limsup \mu_n[A] \;\leq\; \mu[A].$$

(vi) For any open set $O \subseteq S$,

$$\liminf \mu_n[O] \;\geq\; \mu[O].$$

(v) For any $B \in \mathcal{B}(S)$ such that $\mu[\partial B] = 0$,

$$\lim \mu_n[B] \;=\; \mu[B].$$